

# HARMONIC SPHERES IN OUTER SYMMETRIC SPACES, THEIR CANONICAL ELEMENTS AND WEIERSTRASS-TYPE REPRESENTATIONS

N. CORREIA AND R. PACHECO

**ABSTRACT.** Making use of Murakami's classification of outer involutions in a Lie algebra and following the Morse-theoretic approach to harmonic two-spheres in Lie groups introduced by Burstall and Guest, we obtain a new classification of harmonic two-spheres in outer symmetric spaces and a Weierstrass-type representation for such maps. Several examples of harmonic maps into classical outer symmetric spaces are given in terms of meromorphic functions on  $S^2$ .

## 1. INTRODUCTION

The harmonicity of maps  $\varphi$  from a Riemann surface  $M$  into a compact Lie group  $G$  with identity  $e$  amounts to the flatness of one-parameter families of connections. This establishes a correspondence between such maps and certain holomorphic maps  $\Phi$  into the based loop group  $\Omega G$ , the *extended solutions* [17]. Evaluating an extended solution  $\Phi$  at  $\lambda = -1$  we obtain a harmonic map  $\varphi$  into the Lie group. If an extended solution takes values in the group of algebraic loops  $\Omega_{\text{alg}} G$ , the corresponding harmonic map is said to have *finite uniton number*. It is well known that all harmonic maps from the two-sphere into a compact Lie group have finite uniton number [17].

Burstall and Guest [1] have used a method suggested by Morse theory in order to describe harmonic maps with finite uniton number from  $M$  into a compact Lie group  $G$  with trivial centre. One of the main ingredients in that paper is the Bruhat decomposition of the group of algebraic loops  $\Omega_{\text{alg}} G$ . Each piece  $U_\xi$  of the Bruhat decomposition corresponds to an element  $\xi$  in the integer lattice  $\mathfrak{I}(G) = (2\pi)^{-1} \exp^{-1}(e) \cap \mathfrak{t}$  and can be described as the unstable manifold of the energy flow on the Kähler manifold  $\Omega_{\text{alg}} G$ . Each extended solution  $\Phi : M \rightarrow \Omega_{\text{alg}} G$  takes values, off some discrete subset  $D$  of  $M$ , in one of these unstable manifolds  $U_\xi$  and can be deformed, under the gradient flow of the energy, to an extended solution with values in some conjugacy class of a Lie group homomorphism  $\gamma_\xi : S^1 \rightarrow G$ . A normalization procedure allows us to choose  $\xi$  among the *canonical elements* of  $\mathfrak{I}(G)$ ; there are precisely  $2^n$  canonical elements, where  $n = \text{rank}(G)$ , and consequently  $2^n$  classes of harmonic maps. Burstall and Guest [1] introduced also a Weierstrass-type representation for such harmonic maps in terms of meromorphic functions on  $M$ . It is possible to define a similar notion of canonical element for compact Lie groups  $G$  with non-trivial centre [5, 6]. In the present paper, we will not assume any restriction on the centre of  $G$ .

Given an involution  $\sigma$  of  $G$ , the compact symmetric  $G$ -space  $N = G/G^\sigma$ , where  $G^\sigma$  is the subgroup of  $G$  fixed by  $\sigma$ , can be embedded totally geodesically in  $G$  via the corresponding Cartan embedding  $\iota_\sigma$ . Hence harmonic maps into compact symmetric spaces can be interpreted as special harmonic maps into Lie groups. For inner involutions  $\sigma = \text{Ad}(s_0)$ , where  $s_0 \in G$  is the geodesic reflection at some base point  $x_0 \in N$ , the composition of the Cartan embedding with left multiplication by  $s_0$  gives a totally geodesic embedding of  $G/G^\sigma$  in  $G$  as a connected component of  $\sqrt{e}$ . Reciprocally, any connected component of  $\sqrt{e}$  is a compact inner symmetric  $G$ -space. As shown by Burstall and Guest [1], any harmonic map into a connected component of  $\sqrt{e}$  admits an extended solution  $\Phi$  which is invariant under the involution  $I(\Phi)(\lambda) = \Phi(-\lambda)\Phi(-1)^{-1}$ . Off a discrete set,  $\Phi$  takes values in some unstable manifold  $U_\xi$  and can be deformed, under the gradient flow of the energy, to an extended solution with values in some conjugacy class of a Lie group homomorphism  $\gamma_\xi : S^1 \rightarrow G^\sigma$ . An appropriate normalization procedure which preserves both  $I$ -invariance and the underlying

connected component of  $\sqrt{e}$  allows us to choose  $\xi$  among the canonical elements of  $\mathfrak{I}(G^\sigma)$ . As a matter of fact, since  $\sigma$  is inner,  $\text{rank}(G) = \text{rank}(G^\sigma)$  and we have  $\mathfrak{I}(G) = \mathfrak{I}(G^\sigma)$ , that is the canonical elements of  $\mathfrak{I}(G)$  coincide with the canonical elements of  $\mathfrak{I}(G^\sigma)$ . Consequently, if  $G$  has trivial center, we have  $2^n$  classes of harmonic maps with finite uniton number into *all* inner symmetric  $G$ -spaces.

The theory of Burstall and Guest [1] on harmonic two-spheres in compact inner symmetric  $G$ -spaces was extended by Eschenburg, Mare and Quast [8] to outer symmetric spaces as follows: each harmonic map from a two-sphere into an outer symmetric space  $G/G^\sigma$ , with outer involution  $\sigma$ , corresponds to an extended solution  $\Phi$  which is invariant under a certain involution  $T_\sigma$  induced by  $\sigma$  on  $\Omega G$  (see also [11]);  $\Phi$  takes values in some unstable manifold  $U_\xi$ , off some discrete set; under the gradient flow of the energy any such invariant extended solution is deformed to an extended solution with values in some conjugacy class of a Lie group homomorphism  $\gamma_\xi : S^1 \rightarrow G^\sigma$ ; applying the normalization procedure of extended solutions introduced by Burstall and Guest for Lie groups,  $\xi$  can be chosen among the canonical elements of  $\mathfrak{I}(G^\sigma) \subsetneq \mathfrak{I}(G)$ ; if  $G$  has trivial centre, there are precisely  $2^k$  canonical homomorphisms, where  $k = \text{rank}(G^\sigma) < \text{rank}(G)$ ; hence there are *at most*  $2^k$  classes of harmonic two-spheres in  $G/G^\sigma$  if  $G$  has trivial centre.

In the present paper, we will establish a more accurate classification of harmonic maps from a two-sphere into compact outer symmetric spaces. This classification takes into consideration the following crucial facts concerning extended solutions associated to harmonic maps into outer symmetric spaces: although any harmonic map from a two-sphere into an outer symmetric space  $G/G^\sigma$  admits a  $T_\sigma$ -invariant extended solution, not all  $T_\sigma$ -invariant extended solutions correspond to harmonic maps into  $G/G^\sigma$ ; on the other hand, the Burstall and Guest's normalization procedure does not necessarily preserve  $T_\sigma$ -invariance.

Our strategy is the following. The existence of outer involutions of a simple Lie algebra  $\mathfrak{g}$  depends on the existence of non-trivial involutions of the Dynkin diagram of  $\mathfrak{g}^\mathbb{C}$  [2, 8, 12, 14]. More precisely, if  $\varrho$  is a non-trivial involution of the Dynkin diagram of  $\mathfrak{g}^\mathbb{C}$ , then it induces an outer involution  $\sigma_\varrho$  of  $\mathfrak{g}^\mathbb{C}$ , which we call the *fundamental outer involution*, and, as shown by Murakami [14], all the other outer involutions are, up to conjugation, of the form  $\sigma_{\varrho,i} := \text{Ad exp } \pi \zeta_i \circ \sigma_\varrho$  where each  $\zeta_i$  is a certain element in the integer lattice  $\mathfrak{I}(G^{\sigma_\varrho})$ . Each connected component of  $P^{\sigma_\varrho} = \{g \in G \mid \sigma_\varrho(g) = g^{-1}\}$  is a compact outer symmetric  $G$ -space associated to some involution  $\sigma_\varrho$  or  $\sigma_{\varrho,i}$ ; reciprocally, any outer symmetric space  $G/G^\sigma$ , with  $\sigma$  equal to  $\sigma_\varrho$  or  $\sigma_{\varrho,i}$ , can be totally geodesically embedded in the Lie group  $G$  as a connected component of  $P^{\sigma_\varrho}$  (see Proposition 6). As shown in Section 4.2, any harmonic map  $\varphi$  into a connected component  $N$  of  $P^{\sigma_\varrho}$  admits a  $T_{\sigma_\varrho}$ -invariant extended solution  $\Phi$ ; off a discrete set,  $\Phi$  takes values in some unstable manifold  $U_\xi$ . In Section 4.2.2 we introduce an appropriate normalization procedure in order to obtain from  $\Phi$  a *normalized* extended solution  $\tilde{\Phi}$  with values in some unstable manifold  $U_\zeta$  such that:  $\zeta$  is a canonical element of  $\mathfrak{I}(G^{\sigma_\varrho})$ ;  $\tilde{\Phi}$  is  $T_\tau$ -invariant, where  $\tau$  is the outer involution given by  $\tau = \text{Ad exp } \pi(\xi - \zeta) \circ \sigma_\varrho$ ;  $\tilde{\Phi}(-1)$  takes values in some connected component of  $P^{\sigma_\varrho}$  which is an isometric copy of  $N$  completely determined by  $\zeta$  and  $\tau$ ; moreover,  $\tilde{\Phi}(-1)$  coincides with  $\varphi$  up to isometry. Hence, we obtain a classification of harmonic maps of finite uniton number from  $M$  into outer symmetric  $G$ -spaces in terms of the pairs  $(\zeta, \tau)$ .

Dorfmeister, Pedit and Wu [7] have introduced a general scheme for constructing harmonic maps from a Riemann surface into a compact symmetric space from holomorphic data, in which the harmonic map equation reduces to a linear ODE similar to the classical Weierstrass representation of minimal surfaces. Burstall and Guest [1] made this scheme more explicit for the case  $M = S^2$  by establishing a “Weierstrass formula” for harmonic maps with finite uniton number into Lie groups and their inner symmetric spaces. In Theorem 17 we establish a version of this formula to outer symmetric spaces, which allows us to describe the corresponding  $T_\sigma$ -invariant extended solutions in terms of meromorphic functions on  $M$ . For normalized extended solutions and “low uniton number”, such descriptions are easier to obtain. In Section 5 we give several explicit examples of harmonic maps from the two-sphere into classical outer symmetric spaces: Theorem 20 interprets old results by Calabi [3] and Eells and Wood [9] concerning harmonic spheres in real projective spaces  $\mathbb{R}P^{2n-1}$  in view of our classification; harmonic two-spheres into the real Grassmannian  $G_3(\mathbb{R}^6)$  are studied in detail; we show that all harmonic two spheres into the *Wu manifold*  $SU(3)/SO(3)$  can be obtained explicitly by choosing two

meromorphic functions on  $S^2$  and then performing a finite number of algebraic operations, in agreement with the explicit constructions established by H. Ma in [13].

## 2. GROUPS OF ALGEBRAIC LOOPS

For completeness, in this section we recall some fundamental facts concerning the structure of the group of algebraic loops in a compact Lie group. Further details can be found in [1, 4, 15].

Let  $G$  be a compact matrix semisimple Lie group with Lie algebra  $\mathfrak{g}$  and identity  $e$ . Denote the *free* and *based* loop groups of  $G$  by  $\Lambda G$  and  $\Omega G$ , respectively, whereas  $\Lambda_+ G^\mathbb{C}$  stands for the subgroup of  $\Lambda G^\mathbb{C}$  consisting of loops  $\gamma : S^1 \rightarrow G^\mathbb{C}$  which extend holomorphically to the unitary disc  $|\lambda| < 1$ .

Fix a maximal torus  $T$  of  $G$  with Lie algebra  $\mathfrak{t} \subset \mathfrak{g}$ . Let  $\Delta \subset \mathfrak{t}^*$  be the corresponding set of roots, where  $\mathfrak{i} := \sqrt{-1}$ , and, for each  $\alpha \in \Delta$ , denote by  $\mathfrak{g}_\alpha$  the corresponding root space. Choose a fundamental Weyl chamber  $\mathcal{W}$  in  $\mathfrak{t}$ , which corresponds to fix a positive root system  $\Delta^+$ . The intersection  $\mathcal{J}'(G) := \mathcal{J}(G) \cap \mathcal{W}$  parameterizes the conjugacy classes of homomorphisms  $S^1 \rightarrow G$ . More precisely,  $\text{Hom}(S^1, G)$  is the disjoint union of  $\Omega_\xi(G)$ , with  $\xi \in \mathcal{J}'(G)$ , where  $\Omega_\xi(G)$  is the conjugacy class of homomorphisms which contains  $\gamma_\xi(\lambda) = \exp(-\mathfrak{i} \ln(\lambda)\xi)$ .

The *Bruhat decomposition* states that the subgroup of algebraic based loops  $\Omega_{\text{alg}} G$  is the disjoint union of the orbits  $\Lambda_{\text{alg}}^+ G^\mathbb{C} \cdot \gamma_\xi$ , with  $\xi \in \mathcal{J}'(G)$ , where  $\cdot$  denotes the dressing action of  $\Lambda_+ G^\mathbb{C}$  on  $\Omega G$  induced by the *Iwasawa decomposition*  $\Lambda G^\mathbb{C} \cong \Omega G \times \Lambda_+ G^\mathbb{C}$ . According to the Morse theoretic interpretation [1, 15] of the Bruhat decomposition, for each  $\xi \in \mathcal{J}'(G)$ ,  $U_\xi(G) := \Lambda_{\text{alg}}^+ G^\mathbb{C} \cdot \gamma_\xi$  is the unstable manifold of  $\Omega_\xi(G)$  under the flow induced by the energy gradient vector field  $-\nabla E$ : each  $\gamma \in U_\xi(G)$  flows to some homomorphism  $u_\xi(\gamma)$  in  $\Omega_\xi(G)$ .

In [1], the authors proved that, for each  $\xi \in \mathcal{J}'(G)$ , the critical manifold  $\Omega_\xi(G)$  is a complex homogeneous space of  $G^\mathbb{C}$  and the unstable manifold  $U_\xi(G)$  is a complex homogeneous space of the group  $\Lambda_{\text{alg}}^+ G^\mathbb{C}$ . Moreover,  $U_\xi(G)$  carries a structure of holomorphic vector bundle over  $\Omega_\xi(G)$  and the bundle map  $u_\xi : U_\xi(G) \rightarrow \Omega_\xi(G)$  is precisely the natural projection given by  $[\gamma] \mapsto [\gamma(0)]$ .

Define a partial order  $\preceq$  over  $\mathcal{J}'(G)$  as follows:  $\xi \preceq \xi'$  if  $\mathfrak{p}_i^\xi \subset \mathfrak{p}_i^{\xi'}$  for all  $i \geq 0$ , where  $\mathfrak{p}_i^\xi = \sum_{j \leq i} \mathfrak{g}_j^\xi$  and  $\mathfrak{g}_j^\xi$  is the  $j$ -eigenspace of  $\text{ad} \xi$ . As shown in [4], one can define a  $\Lambda_{\text{alg}}^+ G^\mathbb{C}$ -invariant fibre bundle morphism  $\mathcal{U}_{\xi, \xi'} : U_\xi(G) \rightarrow U_{\xi'}(G)$  by

$$\mathcal{U}_{\xi, \xi'}(\Psi \cdot \gamma_\xi) = \Psi \cdot \gamma_{\xi'}, \quad \Psi \in \Lambda_{\text{alg}}^+ G^\mathbb{C},$$

whenever  $\xi \preceq \xi'$ . Since the holomorphic structures on  $U_\xi(G)$  and  $U_{\xi'}(G)$  are induced by the holomorphic structure on  $\Lambda_{\text{alg}}^+ G^\mathbb{C}$ , the fibre-bundle morphism  $\mathcal{U}_{\xi, \xi'}$  is holomorphic.

## 3. HARMONIC SPHERES IN LIE GROUPS

Harmonic maps from the two-sphere  $S^2$  into a compact matrix Lie group  $G$  can be classified in terms of certain pieces of the Bruhat decomposition of  $\Omega_{\text{alg}} G$ . Next we recall briefly this theory from [1, 4, 5, 6].

**3.1. Extended Solutions.** Let  $M$  be a simply-connected Riemann surface,  $\varphi : M \rightarrow G$  be a smooth map and  $\rho : G \rightarrow \text{End}(V)$  a finite representation of  $G$ . Equip  $G$  with a bi-invariant metric. If  $\varphi$  is an harmonic map of *finite uniton number*, it admits an extended solution  $\Phi : M \rightarrow \Omega G$  with  $\Phi(M) \subseteq \Omega_{\text{alg}} G$  and  $\varphi = \Phi_{-1}$ . In this case, we can write  $\rho \circ \Phi = \sum_{i=r}^s \zeta_i \lambda^i$  for some  $r \leq s \in \mathbb{Z}$ . The number  $s - r$  is called the *uniton number* of  $\Phi$  with respect to  $\rho$ , and the minimal value of  $s - r$  (with respect to all extended solutions associated to  $\varphi$ ) is called the *uniton number* of  $\varphi$  with respect to  $\rho$  and it is denoted by  $r_\rho(\varphi)$ . As explained in [6], this definition of uniton number of an extended solution with respect to the adjoint representation is twice that of Burstall and Guest [1]. K. Uhlenbeck [17] proved that all harmonic maps from the two-sphere have finite uniton number. For simplicity of exposition, henceforth we will take  $M = S^2$ . However, all our results still hold for harmonic maps of finite uniton number from an arbitrary Riemann surface.

**Theorem 1.** [1] Let  $\Phi : S^2 \rightarrow \Omega_{\text{alg}} G$  be an extended solution. Then there exists some  $\xi \in \mathcal{J}'(G)$ , and some discrete subset  $D$  of  $S^2$ , such that  $\Phi(S^2 \setminus D) \subseteq U_\xi(G)$ .

Given a smooth map  $\Phi : S^2 \setminus D \rightarrow U_\xi(G)$ , consider  $\Psi : S^2 \setminus D \rightarrow \Lambda_{\text{alg}}^+ G^\mathbb{C}$  such that  $\Phi = \Psi \cdot \gamma_\xi$ , that is  $\Psi \gamma_\xi = \Phi b$  for some  $b : S^2 \setminus D \rightarrow \Lambda_{\text{alg}}^+ G^\mathbb{C}$ . Write  $\Psi^{-1} \Psi_z = \sum_{i \geq 0} X'_i \lambda^i$ , and  $\Psi^{-1} \Psi_{\bar{z}} = \sum_{i \geq 0} X''_i \lambda^i$ . Proposition 4.4 in [1] establishes that  $\Phi$  is an extended solution if, and only if,

$$\text{Im} X'_i \subset \mathfrak{p}_{i+1}^\xi, \quad \text{Im} X''_i \subset \mathfrak{p}_i^\xi, \quad (1)$$

where  $\mathfrak{p}_i^\xi = \bigoplus_{j \leq i} \mathfrak{g}_j^\xi$  and  $\mathfrak{g}_j^\xi$  is the  $j$ i-eigenspace of  $\text{ad} \xi$ . The derivative of the harmonic map  $\varphi = \Phi_{-1}$  is given by the following formula.

**Lemma 2.** [4] Let  $\Phi = \Psi \cdot \gamma_\xi : S^2 \rightarrow \Omega_{\text{alg}} G$  be an extended solution and  $\varphi = \Phi_{-1} : S^2 \rightarrow G$  the corresponding harmonic map. Then

$$\varphi^{-1} \varphi_z = -2 \sum_{i \geq 0} b(0) X_i'^{i+1} b(0)^{-1},$$

where  $X_i'^{i+1}$  is the component of  $X'_i$  over  $\mathfrak{g}_{i+1}^\xi$ , with respect to the decomposition  $\mathfrak{g}^\mathbb{C} = \bigoplus \mathfrak{g}_j^\xi$ .

Both the fiber bundle morphism  $\mathcal{U}_{\xi, \xi'} : U_\xi(G) \rightarrow U_{\xi'}(G)$  and the bundle map  $u_\xi : U_\xi(G) \rightarrow \Omega_\xi(G)$  preserve harmonicity.

**Proposition 3.** [1, 4] Let  $\Phi : S^2 \setminus D \rightarrow U_\xi(G)$  be an extended solution. Then

- a)  $u_\xi \circ \Phi : S^2 \setminus D \rightarrow \Omega_\xi(G)$  is an extended solution, with  $\xi \in \mathcal{I}(G)$ ;
- b) for each  $\xi' \in \mathcal{I}(G)$  such that  $\xi \preceq \xi'$ ,  $\mathcal{U}_{\xi, \xi'}(\Phi) = \mathcal{U}_{\xi, \xi'} \circ \Phi : S^2 \setminus D \rightarrow U_{\xi'}(G)$  is an extended solution.

**3.2. Weierstrass representation.** Taking a larger discrete subset if necessary, one obtains a more explicit description for harmonic maps of finite uniton number and their extended solutions as follows.

**Proposition 4.** [1] Let  $\Phi : S^2 \rightarrow \Omega_{\text{alg}} G$  be an extended solution. There exists a discrete set  $D' \supseteq D$  of  $S^2$  such that  $\Phi|_{S^2 \setminus D'} = \exp C \cdot \gamma_\xi$  for some holomorphic vector-valued function  $C : S^2 \setminus D' \rightarrow \mathfrak{u}_\xi^0$ , where  $\mathfrak{u}_\xi^0$  is the finite dimensional nilpotent subalgebra of  $\Lambda_{\text{alg}}^+ \mathfrak{g}^\mathbb{C}$  defined by

$$\mathfrak{u}_\xi^0 = \bigoplus_{0 \leq i < r(\xi)} \lambda^i (\mathfrak{p}_i^\xi)^\perp, \quad (\mathfrak{p}_i^\xi)^\perp = \bigoplus_{i < j \leq r(\xi)} \mathfrak{g}_j^\xi,$$

with  $r(\xi) = \max\{j \mid \mathfrak{g}_j^\xi \neq 0\}$ . Moreover,  $C$  can be extended meromorphically to  $S^2$ .

Conversely, taking account (1) and the well-known formula for the derivative of the exponential map, we see that if  $C : S^2 \rightarrow \mathfrak{u}_\xi^0$  is meromorphic then  $\Phi = \exp C \cdot \gamma_\xi$  is an extended solution if and only if in the expression

$$(\exp C)^{-1} (\exp C)_z = C_z - \frac{1}{2!} (\text{ad} C) C_z + \dots + (-1)^{r(\xi)-1} \frac{1}{r(\xi)!} (\text{ad} C)^{r(\xi)-1} C_z, \quad (2)$$

the coefficient  $\lambda^i$  have zero component in each  $\mathfrak{g}_{i+2}^\xi, \dots, \mathfrak{g}_{r(\xi)}^\xi$ .

**3.3.  $S^1$ -invariant extended solutions.** Extended solutions with values in some  $\Omega_\xi(G)$ , off a discrete subset, are said to be  $S^1$ -invariant. If we take a unitary representation  $\rho : G \rightarrow U(n)$  for some  $n$ , then for any such extended solution  $\Phi$  we have  $\rho \circ \Phi_\lambda = \sum_{i=r}^s \lambda^i \pi_{W_i}$ , where, for each  $i$ ,  $\pi_{W_i}$  is the orthogonal projection onto a complex vector subbundle  $W_i$  of  $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$  and  $\underline{\mathbb{C}}^n = \bigoplus_{i=r}^s W_i$  is an orthogonal direct sum decomposition. Set  $A_i = \bigoplus_{j \leq i} W_j$  so that

$$\{0\} \subset A_r \subset \dots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \dots \subset A_s = \underline{\mathbb{C}}^n.$$

The harmonicity condition amounts to the following conditions on this flag: for each  $i$ ,  $A_i$  is a holomorphic subbundle of  $\underline{\mathbb{C}}^n$ ; the flag is *superhorizontal*, in the sense that, for each  $i$ , we have  $\partial A_i \subseteq A_{i+1}$ , that is, given any section  $s$  of  $A_i$  then  $\frac{\partial s}{\partial z}$  is a section of  $A_{i+1}$  for any local complex coordinate  $z$  of  $S^2$ .

**3.4. Normalization of harmonic maps.** Let  $\Delta_0 := \{\alpha_1, \dots, \alpha_r\} \subset \Delta^+$  be the basis of positive simple roots, with dual basis  $\{H_1, \dots, H_r\} \subset \mathfrak{t}$ , that is  $\alpha_i(H_j) = \delta_{ij}$ , where  $r = \text{rank}(\mathfrak{g})$ . Given  $\xi = \sum n_i H_i$  and  $\xi' = \sum n'_i H_i$  in  $\mathcal{J}'(G)$ , we have  $n_i, n'_i \geq 0$  and observe that  $\xi \preceq \xi'$  if and only if  $n'_i \leq n_i$  for all  $i$ . For each  $I \subseteq \{1, \dots, r\}$ , define the cone  $\mathfrak{C}_I = \left\{ \sum_{i=1}^r n_i H_i \mid n_i \geq 0, n_j = 0 \text{ iff } j \notin I \right\}$ .

**Definition 1.** [6] Let  $\xi \in \mathcal{J}'(G) \cap \mathfrak{C}_I$ . We say that  $\xi$  is a *I-canonical element* of  $G$  with respect to  $\mathcal{W}$  if it is a maximal element of  $(\mathcal{J}'(G) \cap \mathfrak{C}_I, \preceq)$ , that is: if  $\xi \preceq \xi'$  and  $\xi' \in \mathcal{J}'(G) \cap \mathfrak{C}_I$  then  $\xi = \xi'$ .

When  $G$  has trivial centre, which is the case considered in [1], there exists a unique *I-canonical element*, which is given by  $\xi_I = \sum_{i \in I} H_i$ , for each  $I$ . When  $G$  has non-trivial centre, the *I-canonical elements* of  $G$  were described in [5, 6].

Any harmonic map  $\varphi : S^2 \rightarrow G$  admits a *normalized extended solution*, that is, an extended solution  $\Phi$  taking values in  $U_\xi(G)$ , off some discrete set, for some canonical element  $\xi$ . This is a consequence of the following generalization of Theorem 4.5 in [1].

**Theorem 5.** [4] Let  $\Phi : S^2 \setminus D \rightarrow U_\xi(G)$  be an extended solution. Take  $\xi' \in \mathcal{J}'(G)$  such that  $\xi \preceq \xi'$  and  $\mathfrak{g}_0^\xi = \mathfrak{g}_0^{\xi'}$ . Then  $\gamma^{-1} := \mathcal{U}_{\xi, \xi'}(\Phi)$  is a constant loop in  $\Omega_{\text{alg}} G$  and  $\gamma\Phi : S^2 \setminus D \rightarrow U_{\xi'}(G)$ .

The uniton number of a normalized extended solution  $\Phi : S^2 \setminus D \rightarrow U_\xi(G)$  can be computed with respect to any irreducible  $n$ -dimensional representation  $\rho : G \rightarrow \text{End}(V)$  with highest weight  $\omega^*$  and lowest weight  $\varpi^*$  as follows [6]:  $r_\rho(\xi) := \omega^*(\xi) - \varpi^*(\xi)$ .

#### 4. HARMONIC SPHERES IN OUTER SYMMETRIC SPACES

In the following sections we will establish our classification of harmonic maps from  $S^2$  into compact outer symmetric spaces and establish a Weierstrass formula for such harmonic maps. These will allow us to produce some explicit examples of harmonic maps from two-spheres into outer symmetric spaces from meromorphic data.

As we have referred in Section 1, although any harmonic map from a two-sphere into an outer symmetric space  $G/K$  admits a  $T_\sigma$ -invariant extended solution, not all  $T_\sigma$ -invariant extended solutions correspond to harmonic maps into  $G/K$ ; by Proposition 6 and Theorem 10 below, they correspond to a harmonic map into some possibly different outer symmetric space  $G/K'$  (compare Theorem 20 with Theorem 23 for an example where this happens).

**4.1. Symmetric  $G$ -spaces and Cartan embeddings.** Let  $N = G/K$  be a symmetric space, where  $K$  is the isotropy subgroup at the base point  $x_0 \in N$ , and let  $\sigma : G \rightarrow G$  be the corresponding involution: we have  $G_0^\sigma \subseteq K \subseteq G^\sigma$ , where  $G^\sigma$  is the subgroup fixed by  $\sigma$  and  $G_0^\sigma$  denotes its connected component of the identity. We assume that  $N$  is a *bottom space*, i.e.  $K = G^\sigma$ . Let  $\mathfrak{g} = \mathfrak{k}_\sigma \oplus \mathfrak{m}_\sigma$  be the  $\pm 1$ -eigenspace decomposition associated to the involution  $\sigma$ , where  $\mathfrak{k}_\sigma$  is the Lie algebra of  $K$ . Consider the (totally geodesic) *Cartan embedding*  $\iota_\sigma : N \hookrightarrow G$  defined by  $\iota_\sigma(g \cdot x_0) = g\sigma(g^{-1})$ . The image of the Cartan embedding is precisely the connected component  $P_e^\sigma$  of  $P^\sigma := \{g \in G \mid \sigma(g) = g^{-1}\}$  containing the identity  $e$  of the group  $G$ . Observe that, given  $\xi \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$ , then  $\exp(\pi\xi) \in P^\sigma$ . We denote by  $P_\xi^\sigma$  the connected component of  $P^\sigma$  containing  $\exp(\pi\xi)$ .

**Proposition 6.** Given  $\xi \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$ , we have the following.

- a)  $G$  acts transitively on  $P_\xi^\sigma$  as follows: for  $g \in G$  and  $h \in P_\xi^\sigma$ ,

$$g \cdot_\sigma h = gh\sigma(g^{-1}). \quad (3)$$

- b)  $P_\xi^\sigma$  is a bottom symmetric  $G$ -space totally geodesically embedded in  $G$  with involution

$$\tau = \text{Ad}(\exp \pi\xi) \circ \sigma. \quad (4)$$

- c) For any other  $\xi' \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$  we have  $\exp(\pi\xi') \in P^\tau$  and  $P_{\xi'}^\tau = \exp(\pi\xi)P_{\xi-\xi}^\sigma$ .

- d) The  $\pm 1$ -eigenspace decomposition  $\mathfrak{g} = \mathfrak{k}_\tau \oplus \mathfrak{m}_\tau$  associated to the symmetric  $G$ -space  $P_\xi^\sigma$  at the fixed point  $\exp(\pi\xi) \in P_\xi^\sigma$  is given by

$$\mathfrak{k}_\tau^\mathbb{C} = \bigoplus \mathfrak{g}_{2i}^\xi \cap \mathfrak{k}_\sigma^\mathbb{C} \oplus \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{m}_\sigma^\mathbb{C} \quad (5)$$

$$\mathfrak{m}_\tau^\mathbb{C} = \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{k}_\sigma^\mathbb{C} \oplus \bigoplus \mathfrak{g}_{2i}^\xi \cap \mathfrak{m}_\sigma^\mathbb{C}. \quad (6)$$

*Proof.* Take  $h \in P^\sigma$ . We have

$$\sigma(g \cdot_\sigma h) = \sigma(gh\sigma(g^{-1})) = \sigma(g)h^{-1}g^{-1} = (gh\sigma(g^{-1}))^{-1} = (g \cdot_\sigma h)^{-1}.$$

Then  $g \cdot_\sigma h \in P^\sigma$  and we have a continuous action of  $G$  on  $P^\sigma$ . Since  $G$  is connected, this action induces an action of  $G$  on each connected component of  $P^\sigma$ . Since  $g \cdot_\sigma e = g\sigma(g^{-1}) = \iota_\sigma(g \cdot x_0)$  and  $\iota_\sigma(N) = P_e^\sigma$ , the action  $\cdot_\sigma$  of  $G$  on  $P_e^\sigma$  is transitive.

Take  $\xi \in \mathfrak{I}(G) \cap \mathfrak{k}_\sigma$ , so that  $\sigma(\xi) = \xi$  and  $\exp 2\pi\xi = e$ . Consider the involution  $\tau$  defined by (4). If  $g \in P^\sigma$ , then

$$\tau(\exp(\pi\xi)g) = \exp(\pi\xi)\sigma(\exp(\pi\xi)g)\exp(\pi\xi) = \sigma(g)\exp(\pi\xi) = (\exp(\pi\xi)g)^{-1},$$

which means that  $\exp(\pi\xi)g \in P^\tau$ . Reciprocally, if  $\exp(\pi\xi)g \in P^\tau$ , one can check similarly that  $g \in P^\sigma$ . Hence  $P^\tau = \exp(\pi\xi)P^\sigma$ . In particular, by continuity,  $P_{\xi'}^\tau = \exp(\pi\xi)P_{\xi'-\xi}^\sigma$  for any other  $\xi' \in \mathfrak{I}(G)$  with  $\sigma(\xi') = \xi'$ .

Reversing the rules of  $\sigma = \text{Ad}(\exp \pi\xi) \circ \tau$  and  $\tau$ , we also have  $P_\xi^\sigma = \exp(\pi\xi)P_e^\tau$ . Since  $G$  acts transitively on  $P_e^\tau$ , for each  $h \in P_\xi^\sigma$  there exists  $g \in G$  such that

$$h = \exp(\pi\xi)(g \cdot_\tau e) = (\exp(\pi\xi)g) \cdot_\sigma \exp(\pi\xi).$$

This shows that  $G$  also acts transitively on  $P_\xi^\sigma$ . The isotropy subgroup at  $\exp(\pi\xi)$  consists of those elements  $g$  of  $G$  satisfying  $g \exp(\pi\xi)\sigma(g^{-1}) = \exp(\pi\xi)$ , that is those elements  $g$  of  $G$  which are fixed by  $\tau$ :

$$\exp(\pi\xi)\sigma(g)\exp(\pi\xi) = g. \quad (7)$$

Hence  $P_\xi^\sigma \cong G/G^\tau$ , which is a bottom symmetric  $G$ -space with involution  $\tau$ . Since  $P_e^\tau \subset G$  totally geodesically and  $P_\xi^\sigma$  is the image of  $P_e^\tau$  under an isometry (left multiplication by  $\exp \pi\xi$ ), then  $P_\xi^\sigma \subset G$  totally geodesically.

Differentiating (7) at the identity we get  $\mathfrak{k}_\tau = \{X \in \mathfrak{g} \mid X = \text{Ad}(\exp \pi\xi) \circ \sigma(X)\}$ . Taking account of the formula  $\text{Ad}(\exp(\pi\xi)) = e^{\pi \text{ad} \xi}$  and that  $\sigma$  commutes with  $\text{ad} \xi$ , we obtain (5); and (6) follows similarly.  $\square$

**4.1.1. Outer symmetric spaces.** The existence of outer involutions of a simple Lie algebra  $\mathfrak{g}$  depends on the existence of non-trivial involutions of the Dynkin diagram of  $\mathfrak{g}^\mathbb{C}$  [2, 8, 12, 14]. Fix a maximal abelian subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$  and a Weyl chamber  $\mathcal{W}$  in  $\mathfrak{t}$ , which amounts to fix a system of positive simple roots  $\Delta_0 = \{\alpha_1, \dots, \alpha_r\}$ , where  $r = \text{rank}(\mathfrak{g})$ . Let  $\varrho$  be a non-trivial involution of the Dynkin diagram and  $\sigma_\varrho$  the *fundamental outer involution* associated to  $\varrho$  [2, 14]. The (local isometry classes of) outer symmetric spaces of compact type associated to involutions of the form  $\sigma_\varrho$  are precisely

$$SU(2n)/Sp(n), SU(2n+1)/SO(2n+1), E_6/F_4 \text{ and the real projective spaces } \mathbb{R}P^{2n-1}.$$

These spaces are called the *fundamental outer symmetric spaces*. The remaining classes of outer involutions are obtained as follows [1, 14].

Let  $\mathfrak{g} = \mathfrak{k}_\varrho \oplus \mathfrak{m}_\varrho$  be the corresponding  $\pm 1$ -eigenspace decomposition of  $\mathfrak{g}$ . As shown in Proposition 3.20 of [2], the Lie subalgebra  $\mathfrak{k}_\varrho$  is simple and the orthogonal projection of  $\Delta_0$  onto  $\mathfrak{k}_\varrho$ ,  $\pi_{\mathfrak{k}_\varrho}(\Delta_0)$ , is a basis of positive simple roots of  $\mathfrak{k}_\varrho$  associated to the maximal abelian subalgebra  $\mathfrak{t}_{\mathfrak{k}_\varrho} := \mathfrak{t} \cap \mathfrak{k}_\varrho$ . Consider the split  $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}_\varrho} \oplus \mathfrak{t}_{\mathfrak{m}_\varrho}$  with respect to  $\mathfrak{g} = \mathfrak{k}_\varrho \oplus \mathfrak{m}_\varrho$ . Set  $s = r - k$ , where  $k = \text{rank}(\mathfrak{k}_\varrho)$ . We can label the basis  $\Delta_0$  in order to get the following relations:  $\varrho(\alpha_j) = \alpha_j$  for  $1 \leq j \leq k - s$  and  $\varrho(\alpha_j) = \alpha_{s+j}$  for  $k - s + 1 \leq j \leq k$ . Let  $\pi_{\mathfrak{k}_\varrho}$  be the orthogonal projection of  $\mathfrak{t}$  onto  $\mathfrak{t}_{\mathfrak{k}_\varrho}$ , that is  $\pi_{\mathfrak{k}_\varrho}(H) = \frac{1}{2}(H + \sigma_\varrho(H))$  for all  $H \in \mathfrak{t}$ . Set  $\pi_{\mathfrak{k}_\varrho}(\Delta_0) = \{\beta_1, \dots, \beta_k\}$ , with

$$\beta_j = \begin{cases} \alpha_j & \text{for } 1 \leq j \leq k - s \\ \frac{1}{2}(\alpha_j + \alpha_{j+s}) & \text{for } k - s + 1 \leq j \leq k \end{cases}. \quad (8)$$

This is a basis of  $\mathfrak{it}_{\mathfrak{t}_\varrho}^*$  with dual basis  $\{\zeta_1, \dots, \zeta_k\}$  given by

$$\zeta_j = \begin{cases} H_j & \text{for } 1 \leq j \leq k-s \\ H_j + H_{j+s} & \text{for } k-s+1 \leq j \leq k \end{cases} \quad (9)$$

**Theorem 7.** [14] Let  $\varrho$  be an involution of the Dynkin diagram of  $\mathfrak{g}$ . Let

$$\omega = \sum_{j=1}^{k-s} n_j \beta_j + \sum_{j=k-s+1}^k n'_j \beta_j$$

be the highest root of  $\mathfrak{t}_\varrho$  with respect to  $\pi_{\mathfrak{t}_\varrho}(\Delta_0) = \{\beta_1, \dots, \beta_k\}$ , defined as in (8). Given  $i$  such that  $n_i = 1$  or 2, define an involution  $\sigma_{\varrho, i}$  by

$$\sigma_{\varrho, i} = \text{Ad}(\exp \pi \zeta_i) \circ \sigma_\varrho. \quad (10)$$

Then any outer involution of  $\mathfrak{g}$  is conjugate in  $\mathfrak{Aut}(\mathfrak{g})$ , the group of automorphisms of  $\mathfrak{g}$ , to some  $\sigma_\varrho$  or  $\sigma_{\varrho, i}$ . In particular, there are at most  $k-s+1$  conjugacy classes of outer involutions.

The list of all (local isometry classes of) irreducible outer symmetric spaces of compact type is shown in Table 1 (cf. [2, 8, 12]).

$G/K$	$\text{rank}(G)$	$\text{rank}(K)$	$\text{rank}(G/K)$	$\dim(G/K)$
$SU(2n)/SO(2n)$	$2n-1$	$n$	$2n-1$	$(2n-1)(n+1)$
$SU(2n+1)/SO(2n+1)$	$2n$	$n$	$2n$	$n(2n+3)$
$SU(2n)/Sp(n)$	$2n-1$	$n$	$n-1$	$(n-1)(2n+1)$
$G_p(\mathbb{R}^{2n})$ ( $p$ odd $\leq n$ )	$n$	$n-1$	$p$	$p(2n-p)$
$E_6/Sp(4)$	6	4	6	42
$E_6/F_4$	6	4	2	26

TABLE 1. Irreducible outer symmetric spaces.

Given an outer involution  $\sigma$  of the form  $\sigma_{\varrho, i}$  or  $\sigma_\varrho$  and its  $\pm 1$ -eigenspace decomposition  $\mathfrak{g} = \mathfrak{k}_\sigma \oplus \mathfrak{m}_\sigma$ , set  $\mathfrak{t}_{\mathfrak{t}_\sigma} = \mathfrak{t} \cap \mathfrak{k}_\sigma$ , which is a maximal abelian subalgebra of  $\mathfrak{k}_\sigma$ . Following [8], a non-empty intersection of  $\mathfrak{t}_{\mathfrak{t}_\sigma}$  with a Weyl chamber in  $\mathfrak{t}$  is called a *compartment*. Each compartment lies in a Weyl chamber in  $\mathfrak{t}_{\mathfrak{t}_\sigma}$  and the Weyl chambers in  $\mathfrak{t}_{\mathfrak{t}_\sigma}$  can be decomposed into the same number of compartments [8].

The intersection of the integer lattice  $\mathfrak{I}(G)$  with the Weyl chamber  $\mathcal{W}$  in  $\mathfrak{t}$ , which we have denoted by  $\mathfrak{I}'(G)$ , is described in terms of the dual basis  $\{H_1, \dots, H_r\} \subset \mathfrak{t}$ , with  $r = \text{rank}(\mathfrak{g})$ , by

$$\mathfrak{I}'(G) = \left\{ \sum_{i=1}^r n_i H_i \in \mathfrak{I}(G) \mid n_i \in \mathbb{N}_0 \text{ for all } i \right\}.$$

When  $\sigma$  is a fundamental outer involution  $\sigma_\varrho$ , the compartment  $\mathcal{W} \cap \mathfrak{t}_{\mathfrak{t}_\varrho}$  is itself a Weyl chamber in  $\mathfrak{t}_{\mathfrak{t}_\varrho}$ . Then, the intersection of the integer lattice  $\mathfrak{I}(G^{\sigma_\varrho})$  with the Weyl chamber  $\mathcal{W} \cap \mathfrak{t}_{\mathfrak{t}_\varrho}$ , is given by

$$\mathfrak{I}'(G^{\sigma_\varrho}) = \left\{ \sum_{i=1}^k n_i \zeta_i \in \mathfrak{I}(G) \mid n_i \in \mathbb{N}_0 \text{ for all } i \right\} = \mathfrak{I}'(G) \cap \mathfrak{t}_{\mathfrak{t}_\varrho}.$$

**4.1.2. Cartan embeddings of fundamental outer symmetric spaces.** Next we describe those elements  $\xi$  of  $\mathfrak{I}'(G^{\sigma_\varrho})$  for which the connected component  $P_\xi^{\sigma_\varrho}$  of  $P^{\sigma_\varrho}$  containing  $\exp(\pi \xi)$  can be identified with the fundamental outer symmetric  $G$ -space associated to  $\varrho$ . Start by considering the following  $\sigma_\varrho$ -invariant subsets of the root system  $\Delta \subset \mathfrak{it}^*$  of  $\mathfrak{g}$ :

$$\Delta(\mathfrak{k}_\varrho) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{k}_\varrho^\mathbb{C}\}, \quad \Delta(\mathfrak{m}_\varrho) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{m}_\varrho^\mathbb{C}\}, \quad \Delta_\varrho = \Delta \setminus (\Delta(\mathfrak{k}_\varrho) \cup \Delta(\mathfrak{m}_\varrho)). \quad (11)$$

Then

$$\mathfrak{k}_\varrho^\mathbb{C} = \mathfrak{k}_{\mathfrak{t}_\varrho}^\mathbb{C} \oplus \pi_{\mathfrak{k}_\varrho}(\mathfrak{r}_\varrho) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{k}_\varrho)} \mathfrak{g}_\alpha, \quad \mathfrak{m}_\varrho^\mathbb{C} = \mathfrak{k}_{\mathfrak{m}_\varrho}^\mathbb{C} \oplus \pi_{\mathfrak{m}_\varrho}(\mathfrak{r}_\varrho) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{m}_\varrho)} \mathfrak{g}_\alpha,$$

where  $\mathfrak{r}_\varrho = \bigoplus_{\alpha \in \Delta_\varrho} \mathfrak{g}_\alpha$ . Since the involution  $\varrho$  acts on  $\Delta_\varrho$  as a permutation without fixed points, we can fix some subset  $\Delta'_\varrho$  so that  $\Delta_\varrho$  is the disjoint union of  $\Delta'_\varrho$  with  $\varrho(\Delta'_\varrho)$ :

$$\Delta_\varrho = \Delta'_\varrho \sqcup \varrho(\Delta'_\varrho). \quad (12)$$

For each  $\alpha \in \Delta'_\varrho$ ,  $\sigma_\varrho$  restricts to an involution in the subspace  $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)} \subset \mathfrak{r}_\varrho$ . Hence we have the following.

**Lemma 8.** The orthogonal projections of  $\mathfrak{r}_\varrho$  onto  $\mathfrak{k}_\varrho^\mathbb{C}$  and  $\mathfrak{m}_\varrho^\mathbb{C}$  are given by

$$\pi_{\mathfrak{k}_\varrho}(\mathfrak{r}_\varrho) = \bigoplus_{\alpha \in \Delta'_\varrho} \mathfrak{k}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}), \quad \pi_{\mathfrak{m}_\varrho}(\mathfrak{r}_\varrho) = \bigoplus_{\alpha \in \Delta'_\varrho} \mathfrak{m}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}),$$

and, for each  $\alpha \in \Delta'_\varrho$ ,

$$\mathfrak{k}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma_\varrho(\alpha)}) = \{X_\alpha + \sigma_\varrho(X_\alpha) \mid X_\alpha \in \mathfrak{g}_\alpha\}, \quad \mathfrak{m}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma_\varrho(\alpha)}) = \{X_\alpha - \sigma_\varrho(X_\alpha) \mid X_\alpha \in \mathfrak{g}_\alpha\}.$$

In particular,  $\dim \mathfrak{r}_\varrho = 2 \dim \pi_{\mathfrak{k}_\varrho}(\mathfrak{r}_\varrho) = 2 \dim \pi_{\mathfrak{m}_\varrho}(\mathfrak{r}_\varrho)$ .

**Proposition 9.** Consider the dual basis  $\{\zeta_1, \dots, \zeta_k\}$  defined by (9). Given  $\xi \in \mathcal{J}'(G^{\sigma_\varrho})$  with  $\xi = \sum_{i=1}^k n_i \zeta_i$  and  $n_i \geq 0$ , then  $P_\xi^{\sigma_\varrho}$  is a fundamental outer symmetric space with involution (conjugated to)  $\sigma_\varrho$  if and only if  $n_i$  is even for each  $1 \leq i \leq k-s$ .

*Proof.* There is only one class of outer symmetric  $SU(2n+1)$ -spaces and, in this case, the involution  $\varrho$  does not fix any simple root, that is  $k-s=0$ . Hence the result trivially holds for  $N = SU(2n+1)/SO(2n+1)$ .

Next we consider the remaining fundamental outer symmetric spaces, which are precisely the symmetric spaces of *rank-split type* [8], those satisfying  $\Delta(\mathfrak{m}_\varrho) = \emptyset$ . For such symmetric spaces, the reductive symmetric term  $\mathfrak{m}_\varrho$  satisfies  $\mathfrak{m}_\varrho = \mathfrak{k}_{\mathfrak{m}_\varrho} \oplus \pi_{\mathfrak{m}_\varrho}(\mathfrak{r}_\varrho)$ . On the other hand, in view of (6), we have, for  $\tau = \text{Ad}(\exp \pi \xi) \circ \sigma_\varrho$ ,

$$\begin{aligned} \mathfrak{m}_\tau^\mathbb{C} &= \bigoplus \mathfrak{g}_{2i+1}^\xi \cap \mathfrak{k}_\varrho^\mathbb{C} \oplus \bigoplus \mathfrak{g}_{2i}^\xi \cap \mathfrak{m}_\varrho^\mathbb{C} \\ &= \mathfrak{k}_{\mathfrak{m}_\varrho}^\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{k}_\varrho) \cap \Delta_\xi^-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta'_\varrho \cap \Delta_\xi^-} \mathfrak{k}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}) \oplus \bigoplus_{\alpha \in \Delta'_\varrho \cap \Delta_\xi^+} \mathfrak{m}_\varrho^\mathbb{C} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}), \end{aligned}$$

where  $\Delta_\xi^+ := \{\alpha \in \Delta \mid \alpha(\xi) \text{ is even}\}$  and  $\Delta_\xi^- := \{\alpha \in \Delta \mid \alpha(\xi) \text{ is odd}\}$ . Taking into account Lemma 8, from this we see that  $\dim \mathfrak{m}_\tau = \dim \mathfrak{m}_\varrho$  (which means, by Table 1, that  $P_\xi^{\sigma_\varrho}$  is a fundamental outer symmetric space with involution conjugated to  $\sigma_\varrho$ ) if and only if

$$\bigoplus_{\alpha \in \Delta(\mathfrak{k}_\varrho) \cap \Delta_\xi^-} \mathfrak{g}_\alpha = \{0\},$$

which holds if and only if  $\xi = \sum_{i=1}^k n_i \zeta_i$  with  $n_i$  even for each  $1 \leq i \leq k-s$ .  $\square$

**4.2. Harmonic spheres in symmetric  $G$ -spaces.** Given an involution  $\sigma$  on  $G$ , define an involution  $T_\sigma$  on  $\Omega G$  by  $T_\sigma(\gamma)(\lambda) = \sigma(\gamma(-\lambda)\gamma(-1)^{-1})$ . Let  $\Omega^\sigma G$  be the fixed set of  $T_\sigma$ .

**Theorem 10.** [8, 11] Given  $\xi \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$ , any harmonic map  $\varphi : S^2 \rightarrow P_\xi^\sigma \subset G$  admits an  $T_\sigma$ -invariant extended solution  $\Phi : S^2 \rightarrow \Omega^\sigma G$ . Conversely, given an  $T_\sigma$ -invariant extended solution  $\Phi$ , the smooth map  $\varphi = \Phi_{-1}$  from  $S^2$  is harmonic and takes values in some connected component of  $P^\sigma$ .

**Proposition 11.** [8] Given  $\Phi \in U_\xi^\sigma(G) := U_\xi(G) \cap \Omega^\sigma G$ , with  $\xi \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$ , set  $\gamma = u_\xi \circ \Phi$ . Then  $\gamma$  takes values in  $K$ . By continuity,  $\Phi_{-1}$  and  $\gamma(-1)$  take values in the same connected component of  $P^\sigma$ .

Hence, together with Theorems 1 and 10, this implies the following.



**Theorem 12.** Any harmonic map  $\varphi$  from  $S^2$  into a connected component of  $P^\sigma$  admits an extended solution  $\Phi : S^2 \setminus D \rightarrow U_\xi^\sigma(G) := U_\xi(G) \cap \Omega^\sigma G$ , for some  $\xi \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$  and some discrete subset  $D$ . If  $\sigma = \sigma_\varrho$  is the fundamental outer involution, then  $\varphi = \Phi_{-1}$  takes values in  $P_\xi^{\sigma_\varrho}$ .

*Proof.* By Proposition 11,  $\Phi$  and  $\gamma := u_\xi \circ \Phi$  take values in the same connected component of  $P^\sigma$  when evaluated at  $\lambda = -1$ . Since  $\gamma : S^1 \rightarrow G^\sigma$  is a homomorphism,  $\gamma$  is in the  $G^\sigma$ -conjugacy class of  $\gamma_{\xi'}$  for some  $\xi' \in \mathcal{J}'(G^\sigma)$ , where  $G^\sigma$  is the subgroup of  $G$  fixed by  $\sigma$ . Consequently,  $\gamma(-1) = g\gamma_{\xi'}(-1)g^{-1} = g \cdot \sigma \gamma_{\xi'}(-1)$ , for some  $g \in G^\sigma$ , which means that  $\gamma(-1)$  takes values in the connected component  $P_{\xi'}^\sigma$ . On the other hand,  $\gamma$  is in the  $G$ -conjugacy class of  $\gamma_\xi$ , with  $\xi \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$ . If  $\sigma$  is the fundamental outer involution  $\sigma_\varrho$ , then  $\mathcal{J}'(G^\sigma) = \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$ ; and we must have  $\xi = \xi'$ .  $\square$

**Remark 1.** If  $\sigma$  is not a fundamental outer involution, each Weyl chamber  $\mathcal{W}_\sigma$  in  $\mathfrak{k}_\sigma$  can be decomposed into more than one compartment:  $\mathcal{W}_\sigma = C_1 \sqcup \dots \sqcup C_l$ , where  $C_1 = \mathcal{W} \cap \mathfrak{k}_\sigma$  and the remaining compartments are conjugate to  $C_1$  under  $G$  [8], that is, there exists  $g_i \in G$  satisfying  $C_i = \text{Ad}(g_i)(C_1)$  for each  $i$ . Hence, if we have an extended solution  $\Phi : S^2 \setminus D \rightarrow U_\xi^\sigma(G)$  with  $\xi \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma \subset C_1$ , the corresponding harmonic map  $\Phi_{-1}$  takes values in one of the connected components  $P_{g_i \xi g_i^{-1}}^\sigma$ .

4.2.1.  *$\varrho$ -canonical elements.* Let  $I$  be a subset of  $\{1, \dots, k\}$ , with  $k = \text{rank}(\mathfrak{k}_\varrho)$ , and set

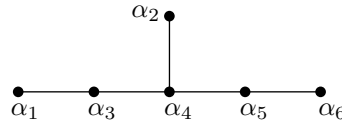
$$\mathfrak{C}_I^\varrho = \left\{ \sum_{i=1}^k n_i \zeta_i \mid n_i \geq 0, n_j = 0 \text{ iff } j \notin I \right\}.$$

Let  $\xi \in \mathcal{J}'(G^{\sigma_\varrho}) \cap \mathfrak{C}_I^\varrho$ . We say that  $\zeta$  is a  *$\varrho$ -canonical element* of  $G$  (with respect to the choice of  $\mathcal{W}$ ) if  $\zeta$  is a maximal element of  $(\mathcal{J}'(G^{\sigma_\varrho}) \cap \mathfrak{C}_I^\varrho, \preceq)$ , that is: if  $\zeta \preceq \zeta'$  and  $\zeta' \in \mathcal{J}'(G^{\sigma_\varrho}) \cap \mathfrak{C}_I^\varrho$  then  $\zeta = \zeta'$ .

**Remark 2.** When  $G$  has trivial centre, the duals  $\zeta_1, \dots, \zeta_k$  belong to the integer lattice. Then, for each  $I$  there exists a unique  $\varrho$ -canonical element, which is given by  $\zeta_I = \sum_{i \in I} \zeta_i$ . In this case, our definition of  $\varrho$ -canonical element coincides with that of  $S$ -canonical element in [8].

Now, consider a fundamental outer involution  $\sigma_\varrho$  and let  $N$  be an associated outer symmetric  $G$ -space, that is,  $N$  corresponds to an involution of  $G$  of the form  $\sigma_\varrho$  or  $\sigma_{\varrho,i}$ , with  $\zeta_i$  in the conditions of Theorem 7. If  $G$  has trivial centre, we certainly have  $\zeta_i \in \mathcal{J}'(G^{\sigma_\varrho})$ . As a matter of fact, as we will see later, in most cases we have  $\zeta_i \in \mathcal{J}'(G^{\sigma_\varrho})$ , whether  $G$  has trivial centre or not, with essentially one exception: for  $G = SU(2n)$  and  $N = SU(2n)/SO(2n)$ . So, we will treat this case separately and assume henceforth that  $\zeta_i \in \mathcal{J}'(G^{\sigma_\varrho})$ .

**Remark 3.** Consider the Dynkin diagram of  $\mathfrak{e}_6$ :



This admits a unique nontrivial involution  $\varrho$ . Let  $\{H_1, \dots, H_6\}$  be the dual basis of  $\Delta_0 = \{\alpha_1, \dots, \alpha_6\}$ . The semi-fundamental basis  $\pi_{\mathfrak{k}_\varrho}(\Delta_0) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$  is given by  $\beta_1 = \alpha_2$ ,  $\beta_2 = \alpha_4$ ,  $\beta_3 = \frac{\alpha_1 + \alpha_6}{2}$  and  $\beta_4 = \frac{\alpha_3 + \alpha_5}{2}$ , whereas the dual basis is given by  $\zeta_1 = H_2$ ,  $\zeta_2 = H_4$ ,  $\zeta_3 = H_1 + H_6$  and  $\zeta_4 = H_3 + H_5$ . Taking account that the elements  $H_i$  are related with the duals  $\eta_i$  of the fundamental weights by

$$[H_i] = \begin{bmatrix} 4/3 & 1 & 5/3 & 2 & 4/3 & 2/3 \\ 1 & 2 & 2 & 3 & 2 & 1 \\ 5/3 & 2 & 10/3 & 4 & 8/3 & 4/3 \\ 2 & 3 & 4 & 6 & 4 & 2 \\ 4/3 & 2 & 8/3 & 4 & 10/3 & 5/3 \\ 2/3 & 1 & 4/3 & 2 & 5/3 & 4/3 \end{bmatrix} [\eta_i],$$

we see that the elements  $\zeta_i$  are in the integer lattice  $\mathcal{J}'(\tilde{E}_6) \subset \mathcal{J}'(E_6)$ , where  $\tilde{E}_6$  is the compact simply connected Lie group with Lie algebra  $\mathfrak{e}_6$ , which has centre  $\mathbb{Z}_3$ , and  $E_6$  is the adjoint group  $\tilde{E}_6/\mathbb{Z}_3$ .

Taking into account Proposition 6, we can identify  $N$  with the connected component  $P_{\zeta_i}^{\sigma_e} = \exp(\pi\zeta_i)P_e^{\sigma_{e,i}}$ , which is a totally geodesic submanifold of  $G$ , via

$$g \cdot x_0 \in N \mapsto \exp(\pi\zeta_i)g\sigma_{e,i}(g^{-1}) \in P_{\zeta_i}^{\sigma_e}. \quad (13)$$

By Theorem 12, each harmonic map  $\varphi : S^2 \rightarrow N \cong P_{\zeta_i}^{\sigma_e}$  admits a  $T_{\sigma_e}$ -invariant extended solution with values, off a discrete set, in some unstable manifold  $U_\xi(G)$ , with  $\xi \in \mathcal{J}'(G^{\sigma_e}) \cap \mathfrak{E}_I^e$ . By Theorem 5, this extended solution can be multiplied on the left by a constant loop in order to get a normalized extended solution with values in some unstable manifold  $U_\zeta(G)$  for some  $\varrho$ -canonical element  $\zeta$ . Hence, if  $G$  has trivial centre, the Bruhat decomposition of  $\Omega_{\text{alg}}G$  gives rise to  $2^k$  classes of harmonic maps into  $P^{\sigma_e}$ , that is  $2^k$  classes of harmonic maps into *all* outer symmetric  $G$ -spaces.

However, the normalization procedure given by Theorem 5 does not preserve  $T_{\sigma_e}$ -invariance, and consequently, as we will see next, normalized extended solutions with values in the same unstable manifold  $U_\zeta(G)$ , for some  $\varrho$ -canonical element  $\zeta$ , correspond in general to harmonic maps into different outer symmetric  $G$ -spaces. Hence the classification of harmonic two-spheres into outer symmetric  $G$ -spaces in terms of  $\varrho$ -canonical elements is manifestly unsatisfactory since it does not distinguish the underlying symmetric space. In the following sections we overcome this weakness by establishing a classification of all such harmonic maps in terms of pairs  $(\zeta, \sigma)$ , where  $\zeta$  is a  $\varrho$ -canonical element and  $\sigma$  an outer involution of  $G$ .

**4.2.2. Normalization of  $T_\sigma$ -invariant extended solutions.** Let  $\sigma$  be an outer involution of  $G$ . The fibre bundle morphisms  $\mathcal{U}_{\xi, \xi'}$  preserve  $T_\sigma$ -invariance:

**Proposition 13.** If  $\xi \preceq \xi'$  and  $\xi, \xi' \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$ , then  $\mathcal{U}_{\xi, \xi'}(U_\xi^\sigma(G)) \subset U_{\xi'}^\sigma(G)$ .

*Proof.* For  $\Phi \in U_\xi^\sigma(G)$ , write  $\Phi = \Psi \cdot \gamma_\xi$  for some  $\Psi \in \Lambda_{\text{alg}}^+ G^\mathbb{C}$ . If  $\Phi$  is  $T_\sigma$ -invariant we have  $\Psi(\lambda) \cdot \gamma_\xi = \sigma(\Psi(-\lambda)) \cdot \gamma_\xi$ . Consequently, we also have  $\Psi(\lambda) \cdot \gamma_{\xi'} = \sigma(\Psi(-\lambda)) \cdot \gamma_{\xi'}$ , which means in turn that  $\mathcal{U}_{\xi, \xi'}(\Phi) = \Psi \cdot \gamma_{\xi'}^\sigma$  is  $T_\sigma$ -invariant.  $\square$

Hence, if  $\Phi : S^2 \setminus D \rightarrow U_\xi^\sigma(G)$  is an extended solution and  $\xi \preceq \xi'$ , with  $\xi, \xi' \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$ , by Theorem 5 and Proposition 13 we know that  $\gamma^{-1} := \mathcal{U}_{\xi, \xi-\xi'}(\Phi)$  is a constant  $T_\sigma$ -invariant loop if  $\mathfrak{g}_0^\xi = \mathfrak{g}_0^{\xi'}$ . However, in general, the product  $\gamma\Phi$  is not  $T_\sigma$ -invariant.

**Lemma 14.** Assume that  $\gamma^{-1}, \Phi \in \Omega^\sigma G$  and  $\gamma(-1) \in P_\xi^\sigma$  for some  $\xi \in \mathcal{J}(G) \cap \mathfrak{k}_\sigma$ . Take  $h \in G$  such that  $\gamma(-1) = h^{-1} \cdot_\sigma \exp(\pi\xi)$ . Then  $h\gamma\Phi h^{-1} \in \Omega^\tau G$ , with  $\tau = \text{Ad}(\exp \pi\xi) \circ \sigma$ .

*Proof.* Since  $\gamma^{-1}, \Phi \in \Omega^\sigma G$ , a simple computation shows that  $T_\sigma(\gamma\Phi) = \gamma(-1)^{-1}\gamma\Phi\gamma(-1)$ . Since  $\gamma(-1) \in P_\xi^\sigma$ , there exists  $h \in G$  such that  $\gamma(-1) = h^{-1} \cdot_\sigma \exp(\pi\xi) = h^{-1} \exp(\pi\xi)\sigma(h)$ . One can check now that  $T_\tau(h\gamma\Phi h^{-1}) = h\gamma\Phi h^{-1}$ .  $\square$

**Proposition 15.** Take  $\xi, \xi' \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$  such that  $\xi \preceq \xi'$ . Let  $\Phi : S^2 \setminus D \rightarrow U_\xi^\sigma(G)$  be a  $T_\sigma$ -invariant extended solution. If  $\gamma^{-1} := \mathcal{U}_{\xi, \xi-\xi'}(\Phi)$  is a constant loop, there exists  $h \in G$  such that  $\tilde{\Phi} := h\gamma\Phi h^{-1}$  takes values in  $U_{\xi'}^\tau(G)$ , with  $\tau = \text{Ad}(\exp \pi(\xi - \xi')) \circ \sigma$ .

Additionally, if  $\sigma$  is the fundamental outer involution  $\sigma_\varrho$ , the harmonic map  $\Phi_{-1}$  takes values in  $P_\xi^\sigma$  and  $\tilde{\Phi}_{-1}$  takes values in  $P_{\xi'}^\tau$ , which implies that  $\Phi_{-1}$  is given, up to isometry, by

$$\exp(\pi(\xi - \xi'))\tilde{\Phi}_{-1} : S^2 \rightarrow P_{\xi'}^\tau.$$

*Proof.* Assume that  $\gamma^{-1} := \mathcal{U}_{\xi, \xi-\xi'}(\Phi) = \Psi \cdot \gamma_{\xi-\xi'}$  is a constant loop. We can write  $\Psi\gamma_{\xi-\xi'} = \gamma^{-1}b$  for some  $b : S^2 \setminus D \rightarrow \Lambda_{\text{alg}}^+ G$ . Then

$$\Phi = \Psi \cdot \gamma_\xi = \Psi \cdot \gamma_{\xi-\xi'}\gamma_{\xi'} = \gamma^{-1}b \cdot \gamma_{\xi'},$$

which implies that  $\gamma\Phi$  takes values in  $U_{\xi'}(G)$ . On the other hand, since  $\gamma^{-1}$  is  $T_\sigma$ -invariant (by Proposition 13),  $\gamma(-1) \in P^\sigma$ .

Take  $\eta \in \mathcal{J}'(G^\sigma)$  and  $h \in G$  such that  $\gamma(-1) \in P_\eta^\sigma$  and  $\gamma(-1) = h^{-1} \cdot_\sigma \exp \pi \eta$ . From Lemma 14, we see that  $\tilde{\Phi} := h\gamma\Phi h^{-1}$  is  $T_\tau$ -invariant. Hence  $\tilde{\Phi}$  takes values in  $U_{\xi'}^\tau(G)$ . Since  $\gamma$  is constant,  $\tilde{\Phi}$  is an extended solution.

If  $\sigma = \sigma_\varrho$ , then  $\mathcal{J}'(G^{\sigma_\varrho}) = \mathcal{J}'(G) \cap \mathfrak{k}_{\sigma_\varrho}$ , which implies that  $\eta = \xi - \xi'$ . The element  $h \in G$  is such that

$$\gamma(-1) = h^{-1} \exp(\pi(\xi - \xi'))\sigma_\varrho(h).$$

On the other hand, since, by Theorem 12,  $\Phi_{-1}$  takes values in  $P_{\xi'}^{\sigma_\varrho}$ , we also have  $\Phi_{-1} = g \exp(\pi\xi)\sigma_\varrho(g^{-1})$  for some lift  $g : S^2 \rightarrow G$ . Hence

$$\begin{aligned} \tilde{\Phi}_{-1} &= h\gamma(-1)\Phi_{-1}h^{-1} = \exp(\pi(\xi - \xi'))\sigma_\varrho(h)g \exp(\pi\xi)\sigma_\varrho(\sigma_\varrho(h)g)^{-1} \\ &= \exp(\pi(\xi - \xi'))(\sigma_\varrho(h)g \cdot_{\sigma_\varrho} \exp \pi\xi) \end{aligned}$$

Hence, in view of Proposition 6,  $\tilde{\Phi}_{-1}$  takes values in  $P_{\xi'}^\tau = \exp(\pi(\xi - \xi'))P_\xi^\sigma$ .  $\square$

Under some conditions on  $\xi \preceq \xi'$ , the morphism  $\mathcal{U}_{\xi, \xi-\xi'}(\Phi)$  is always a constant loop.

**Proposition 16.** Take  $\xi, \xi' \in \mathcal{J}'(G) \cap \mathfrak{k}_\sigma$  such that  $\xi \preceq \xi'$ . Assume that

$$\mathfrak{g}_{2i}^\xi \cap \mathfrak{m}_\sigma^\mathbb{C} \subset \bigoplus_{0 \leq j < 2i} \mathfrak{g}_j^{\xi-\xi'}, \quad \mathfrak{g}_{2i-1}^\xi \cap \mathfrak{k}_\sigma^\mathbb{C} \subset \bigoplus_{0 \leq j < 2i-1} \mathfrak{g}_j^{\xi-\xi'}, \quad (14)$$

for all  $i > 0$ . Then,  $\mathcal{U}_{\xi, \xi-\xi'} : U_\xi^\sigma(G) \rightarrow U_{\xi-\xi'}^\sigma(G)$  transforms  $T_\sigma$ -invariant extended solutions in constant loops.

*Proof.* Given an extended solution  $\Phi : S^2 \setminus D \rightarrow U_\xi^\sigma(G)$ , choose  $\Psi : S^2 \setminus D \rightarrow \Lambda_{\text{alg}}^+ G^\mathbb{C}$  such that  $\Phi = \Psi \cdot \gamma_\xi$  and  $T_\sigma(\Psi) = \Psi$ . Differentiating this we see that

$$\text{Im} \Psi^{-1} \Psi_z \subset \bigoplus_{i \geq 0} \lambda^{2i} \mathfrak{k}_\sigma^\mathbb{C} \oplus \bigoplus_{i \geq 0} \lambda^{2i+1} \mathfrak{m}_\sigma^\mathbb{C}. \quad (15)$$

Write  $\Psi^{-1} \Psi_z = \sum_{r \geq 0} \lambda^r X'_r$ . Since  $\xi \preceq \xi - \xi'$ , by Proposition 3 and Proposition 13,  $\mathcal{U}_{\xi, \xi-\xi'}(\Phi)$  is an extended solution with values in  $U_{\xi-\xi'}^\sigma(G)$ . Hence, taking into account Lemma 2, in order to prove that  $\mathcal{U}_{\xi, \xi-\xi'}(\Phi)$  is constant we only have to check that the component of  $X'_r$  over  $\mathfrak{g}_{r+1}^{\xi-\xi'}$  vanishes for all  $r \geq 0$ .

From (1) and (15) we see that, for  $r = 2i$ ,  $X'_{2i}$  takes values in  $\bigoplus_{j \leq 2i+1} \mathfrak{g}_j^\xi \cap \mathfrak{k}_\sigma^\mathbb{C}$ . But, since  $\xi \preceq \xi - \xi'$  and, by hypothesis, (14) holds, we have

$$\bigoplus_{j \leq 2i+1} \mathfrak{g}_j^\xi \cap \mathfrak{k}_\sigma^\mathbb{C} = \left( \bigoplus_{j \leq 2i} \mathfrak{g}_j^\xi \cap \mathfrak{k}_\sigma^\mathbb{C} \right) \oplus (\mathfrak{g}_{2i+1}^\xi \cap \mathfrak{k}_\sigma^\mathbb{C}) \subset \left( \bigoplus_{j \leq 2i} \mathfrak{g}_j^{\xi-\xi'} \cap \mathfrak{k}_\sigma^\mathbb{C} \right) \oplus \bigoplus_{0 \leq j < 2i+1} \mathfrak{g}_j^{\xi-\xi'}.$$

Hence the component of  $X'_{2i}$  over  $\mathfrak{g}_{2i+1}^{\xi-\xi'}$  vanishes for all  $i \geq 0$ . Similarly, for  $r = 2i - 1$ ,  $X'_{2i-1}$  takes values in  $\bigoplus_{j \leq 2i} \mathfrak{g}_j^\xi \cap \mathfrak{m}_\sigma^\mathbb{C}$ , and we can check that the component of  $X'_{2i-1}$  over  $\mathfrak{g}_{2i}^{\xi-\xi'}$  vanishes for all  $i > 0$ .

Hence  $\gamma^{-1} := \mathcal{U}_{\xi, \xi-\xi'}(\Phi) = \Psi \cdot \gamma_{\xi-\xi'}$  is a constant loop.  $\square$

**Definition 2.** We say that  $\zeta \in \mathcal{J}'(G^{\sigma_\varrho}) \cap \mathfrak{C}_I^\varrho$  is a  $\varrho$ -semi-canonical element if  $\zeta$  is of the form  $\zeta = \sum_{i \in I} n_i \zeta_i$  with  $1 \leq n_i \leq 2m_i$ , where  $m_i$  is the least positive integer which makes  $m_i \zeta_i \in \mathcal{J}'(G^{\sigma_\varrho})$ .

**Corollary 1.** Take  $\xi \in \mathcal{J}'(G^{\sigma_\varrho}) \cap \mathfrak{C}_I^\varrho$ , with  $I \subset \{1, \dots, k\}$ . Let  $\Phi : S^2 \setminus D \rightarrow U_\xi^{\sigma_\varrho}(G)$  be a  $T_{\sigma_\varrho}$ -invariant extended solution, and let  $\varphi : S^2 \rightarrow P_\xi^{\sigma_\varrho}$  be the corresponding harmonic map. Then there exist  $h \in G$ , a constant loop  $\gamma$ , and a  $\varrho$ -semi-canonical  $\zeta$  such that  $\tilde{\Phi} := h\gamma\Phi h^{-1}$  defined on  $S^2 \setminus D$  takes values in  $U_\zeta^{\sigma_\varrho}(G)$ . The harmonic map  $\tilde{\Phi}_{-1}$  takes values in  $P_\zeta^{\sigma_\varrho} = P_\xi^{\sigma_\varrho}$  and coincides with  $\varphi$  up to isometry.

*Proof.* Write  $\xi = \sum_{i \in I} r_i \zeta_i$ , with  $r_i > 0$ . For each  $i \in I$ , let  $n_i$  be the unique integer number in  $\{1, \dots, 2m_i\}$  such that  $n_i = r_i \bmod 2m_i$ . Set  $\zeta = \sum_{i \in I} n_i \zeta_i$ . It is clear that  $\xi \preceq \zeta$  and  $\zeta \in \mathcal{I}'(G^{\sigma_e}) \cap \mathfrak{C}_I^e$ . Observe also that conditions (14) hold automatically for any  $\zeta' \in \mathcal{I}'(G^{\sigma_e}) \cap \mathfrak{C}_I^e$  satisfying  $\xi \preceq \zeta'$ . In particular they hold for  $\zeta' = \zeta$ . Finally, since  $\xi - \zeta = 2 \sum_{i \in I} m_i k_i \zeta_i$  for some nonnegative integer numbers  $k_i$ , then  $\exp \pi(\xi - \zeta) = e$ , and the result follows from Propositions 15 and 16.  $\square$

**4.2.3. Classification of harmonic two-spheres into outer symmetric spaces.** To sum up, in order to classify all harmonic two-spheres into outer symmetric spaces we proceed as follows:

- (1) Start with a fundamental outer involution  $\sigma_e$  and let  $N$  be an outer symmetric  $G$ -space corresponding to an involution of the form  $\sigma_e$  or  $\sigma_{e,i}$  of  $G$ , according to (10), where the element  $\zeta_i$  is in the conditions of Theorem 7. We assume that  $\exp 2\pi\zeta_i = e$ , that is  $\zeta_i \in \mathcal{I}'(G^{\sigma_e})$ . Let  $\varphi : S^2 \rightarrow N$  be an harmonic map and identify  $N$  with  $P_{\zeta_i}^{\sigma_e} = \exp(\pi\zeta_i)P_e^{\sigma_{e,i}}$  via the totally geodesic embedding (13). If  $N$  is the fundamental outer space with involution  $\sigma_e$  we simply identify  $N$  with  $P_e^{\sigma_e}$  via  $\iota_{\sigma_e}$ .
- (2) By Theorem 12,  $\varphi : S^2 \rightarrow N \cong P_{\zeta_i}^{\sigma_e}$  admits a  $T_{\sigma_e}$ -invariant extended solution  $\Phi : S^2 \rightarrow \Omega^{\sigma_e}G$  which takes values, off some discrete subset  $D$ , in some unstable manifold  $U_{\zeta'}^{\sigma_e}(G)$ , with  $\zeta' \in \mathcal{I}'(G^{\sigma_e})$ ; moreover,  $P_{\zeta'}^{\sigma_e} = P_{\zeta_i}^{\sigma_e}$ .
- (3) By Corollary 1, we can assume that  $\zeta'$  is a  $\varrho$ -semi-canonical element in  $\mathcal{I}'(G^{\sigma_e}) \cap \mathfrak{C}_I^e$ . If  $\zeta$  is a  $\varrho$ -canonical element such that  $\zeta' \preceq \zeta$  and  $\mathcal{U}_{\zeta', \zeta' - \zeta}(\Phi)$  is constant, then, taking into account Proposition 15, there exists a  $T_\tau$ -invariant extended solution  $\tilde{\Phi} : S^2 \setminus D \rightarrow U_\zeta^\tau(G)$ , where

$$\tau = \text{Ad}(\exp \pi(\zeta' - \zeta)) \circ \sigma_e, \quad (16)$$

such that the harmonic map  $\varphi$  is given, up to isometry, by  $\tilde{\Phi}_{-1} : S^2 \rightarrow P_\zeta^\tau$ . Here we identify  $N$  with  $P_\zeta^\tau = \exp(\pi(\zeta' - \zeta))P_{\zeta_i}^{\sigma_e}$  via the composition of (13) with the left multiplication by  $\exp(\pi(\zeta' - \zeta))$ .

- (4) By Proposition 16, there always exists a  $\varrho$ -canonical element  $\zeta$  in such conditions.

Hence, we classify harmonic spheres into outer symmetric  $G$ -spaces in terms of pairs  $(\zeta, \tau)$ , where  $\zeta$  is a  $\varrho$ -canonical element and  $\tau$  is an outer involution of the form (16) for some  $\varrho$ -semi-canonical element  $\zeta'$  with  $\zeta' \preceq \zeta$ .

**4.2.4. Weierstrass Representation for  $T_\sigma$ -invariant Extended Solutions.** From (15) and Proposition 4, we obtain the following.

**Theorem 17.** Let  $\Phi : M \rightarrow \Omega_{\text{alg}}^\sigma G$  be an extended solution. There exists a discrete set  $D' \supseteq D$  of  $M$  such that  $\Phi|_{M \setminus D'} = \exp C \cdot \gamma_\xi$  for some holomorphic vector-valued function  $C : M \setminus D' \rightarrow (\mathfrak{u}_\xi^0)_\sigma$ , where  $(\mathfrak{u}_\xi^0)_\sigma$  is the finite dimensional nilpotent subalgebra of  $\Lambda_{\text{alg}}^+ \mathfrak{g}^\mathbb{C}$  defined by

$$(\mathfrak{u}_\xi^0)_\sigma = \bigoplus_{0 \leq 2i < r(\xi)} \lambda^{2i} (\mathfrak{p}_{2i}^\xi)^\perp \cap \mathfrak{k}_\sigma^\mathbb{C} \oplus \bigoplus_{0 \leq 2i+1 < r(\xi)} \lambda^{2i+1} (\mathfrak{p}_{2i+1}^\xi)^\perp \cap \mathfrak{m}_\sigma^\mathbb{C},$$

with  $(\mathfrak{p}_i^\xi)^\perp = \bigoplus_{i < j \leq r(\xi)} \mathfrak{g}_j^\xi$ . Moreover,  $C$  can be extended meromorphically to  $M$ .

## 5. EXAMPLES

Next we will describe explicit examples of harmonic spheres into *classical* outer symmetric spaces.

**5.1. Outer symmetric  $SO(2n)$ -spaces.** For details on the structure of  $\mathfrak{so}(2n)$  see [10]. Consider on  $\mathbb{R}^{2n}$  the standard inner product  $\langle \cdot, \cdot \rangle$  and fix a complex basis  $\mathbf{u} = \{u_1, \dots, u_n, \bar{u}_1, \dots, \bar{u}_n\}$  of  $\mathbb{C}^{2n} = (\mathbb{R}^{2n})^\mathbb{C}$  satisfying

$$\langle u_i, u_j \rangle = 0, \quad \langle u_i, \bar{u}_j \rangle = \delta_{ij}, \quad \text{for all } 1 \leq i, j \leq n.$$

Throughout this section we will denote by  $V_l$  the  $l$ -dimensional isotropic subspace spanned by  $\bar{u}_1, \dots, \bar{u}_l$ .

Set  $E_i = E_{i,i} - E_{n+i, n+i}$ , where  $E_{j,j}$  is a square matrix, with respect to the basis  $\mathbf{u}$ , whose  $(j, j)$ -entry is 1 and all other entries are 0. The complexification  $\mathfrak{t}^\mathbb{C}$  of the algebra  $\mathfrak{t}$  of diagonal matrices  $\sum a_i E_i$ , with  $a_i \in \mathbb{R}$

and  $\sum a_i = 0$ , is a Cartan subalgebra of  $\mathfrak{so}(2n)^\mathbb{C}$ . Let  $\{L_1, \dots, L_n\}$  be the dual basis in  $\mathfrak{t}^*$  of  $\{E_1, \dots, E_n\}$ , that is  $L_i(E_j) = \delta_{ij}$ . The roots of  $\mathfrak{so}(2n)$  are the vectors  $\pm L_i \pm L_j$  and  $\pm L_i \mp L_j$ , with  $i \neq j$  and  $1 \leq i, j \leq n$ .

Consider the endomorphisms

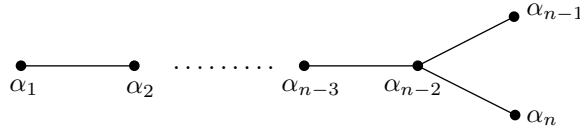
$$X_{i,j} = E_{i,j} - E_{n+j,n+i}, \quad Y_{i,j} = E_{i,n+j} - E_{j,n+i}, \quad Z_{i,j} = E_{n+i,j} - E_{n+j,i}, \quad (17)$$

where  $E_{i,j}$ , with  $i \neq j$ , is a square matrix whose  $(i, j)$ -entry is 1 and all other entries are 0. The root spaces of  $L_i - L_j$ ,  $L_i + L_j$  and  $-L_i - L_j$ , respectively, are generated by the endomorphisms  $X_{i,j}$ ,  $Y_{i,j}$  and  $Z_{i,j}$ , respectively.

Fix the positive root system  $\Delta^+ = \{L_i \pm L_j\}_{i < j}$ . The positive simple roots are  $\alpha_i = L_i - L_{i+1}$ , for  $1 \leq i \leq n-1$ , and  $\alpha_n = L_{n-1} + L_n$ . The vectors of the dual basis  $\{H_1, \dots, H_n\} \subset \mathfrak{t}$  are given by  $H_i = E_1 + E_2 + \dots + E_i$ , for  $1 \leq i \leq n-2$ ,

$$H_{n-1} = \frac{1}{2}(E_1 + E_2 + \dots + E_{n-1} - E_n), \text{ and } H_n = \frac{1}{2}(E_1 + E_2 + \dots + E_{n-1} + E_n).$$

Consider the non-trivial involution  $\varrho$  of the corresponding Dynkin diagram,



This involution fixes  $\alpha_i$  if  $i \leq n-2$  and  $\varrho(\alpha_{n-1}) = \alpha_n$ . The corresponding semi-fundamental basis  $\pi_{\mathfrak{t}_\varrho}(\Delta_0) = \{\beta_1, \dots, \beta_{n-1}\}$  is given by

$$\beta_i = \alpha_i = L_i - L_{i+1}, \text{ if } i \leq n-2, \text{ and } \beta_{n-1} = \frac{1}{2}(\alpha_{n-1} + \alpha_n) = L_{n-1},$$

whereas the dual basis  $\{\zeta_1, \dots, \zeta_{n-1}\}$  is given by  $\zeta_i = E_1 + \dots + E_i$ , with  $i = 1, \dots, n-1$ . Since each  $\zeta_i$  belongs to the integer lattice  $\mathfrak{I}(SO(2n)^{\sigma_e})$ , we have:

**Proposition 18.** The  $\varrho$ -semi-canonical elements of  $SO(2n)$  are precisely the elements  $\zeta = \sum_{i=1}^{n-1} m_i \zeta_i$  such that  $m_i \in \{0, 1, 2\}$  for  $1 \leq i \leq n-1$ .

The fundamental outer symmetric  $SO(2n)$ -space is the real projective space  $\mathbb{R}P^{2n-1}$ , and the associated outer symmetric  $SO(2n)$ -spaces are the real Grassmannians  $G_p(\mathbb{R}^{2n})$  with  $p > 1$  odd.

**5.1.1. Harmonic maps into real projective spaces  $\mathbb{R}P^{2n-1}$ .** Consider as base point the one dimensional real vector space  $V_0$  spanned by  $e_n = (u_n + \bar{u}_n)/\sqrt{2}$  in  $\mathbb{R}^{2n}$ , which establishes an identification of  $\mathbb{R}P^{2n-1}$  with  $SO(2n)/S(O(1)O(2n-1))$ . Denote by  $\pi_{V_0}$  and  $\pi_{V_0}^\perp$  the orthogonal projections onto  $V_0$  and  $V_0^\perp$ , respectively. The fundamental involution is given by  $\sigma_e = \text{Ad}(s_0)$ , where  $s_0 = \pi_{V_0} - \pi_{V_0}^\perp$ . Following the classification procedure established in Section 4.2.3, we start by identifying  $\mathbb{R}P^{2n-1}$  with  $P_e^{\sigma_e}$ .

**Theorem 19.** Each harmonic map  $\varphi : S^2 \rightarrow \mathbb{R}P^{2n-1}$  belongs to one of the following classes:  $(\zeta_l, \sigma_{\varrho,l})$ , with  $1 \leq l \leq n-1$ .

*Proof.* Let  $\zeta$  be a  $\varrho$ -semi-canonical element and write

$$\zeta = \sum_{i \in I_1} \zeta_i + \sum_{i \in I_2} 2\zeta_i \quad (18)$$

for some disjoint subsets  $I_1$  and  $I_2$  of  $\{1, \dots, n-1\}$ . By Proposition 9,  $P_\zeta^{\sigma_e} \cong \mathbb{R}P^{2n-1}$  if and only if either  $I_1 = \emptyset$  or  $I_1 = \{n-1\}$ . Suppose that  $I_1 = \{n-1\}$ . In this case,  $\exp \pi \zeta = \exp \pi \zeta_{n-1} \in P_{\zeta_{n-1}}^{\sigma_e}$ . We claim that  $P_{\zeta_{n-1}}^{\sigma_e}$  is not the connected component of  $P^{\sigma_e}$  containing the identity  $e$ . Write  $\exp \pi \zeta_{n-1} = \pi_V - \pi_V^\perp$ , where  $V$  is the two-dimensional real space spanned by  $e_n$  and  $e_{2n}$ . For each  $g \in P_e^{\sigma_e}$ , since the  $G$ -action  $\cdot_{\sigma_e}$  defined by (3) is transitive, we have  $g = h \cdot_{\sigma_e} e = h s_0 h^{-1} s_0$  for some  $h \in G$ , which means that  $g s_0 = h s_0 h^{-1}$ . In particular, the  $+1$ -eigenspaces of  $g s_0$  must be 1-dimensional. However, a simple computation shows that the  $+1$ -eigenspace of  $\exp(\pi \zeta_{n-1}) s_0$  is 3-dimensional, which establishes our claim.

Then, any harmonic map  $\varphi : S^2 \rightarrow \mathbb{R}P^{2n-1} \cong P_e^{\sigma_e}$  admits a  $T_{\sigma_e}$ -invariant extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta}^{\sigma_e}(SO(2n))$  with  $\zeta$  a  $\varrho$ -semi-canonical element of the form  $\zeta = \sum_{i \in I_2} 2\zeta_i$ . Set  $l = \max I_2$ . Next we check that  $\zeta$  and  $\zeta_l$  satisfy the conditions of Proposition 16, with  $\xi = \zeta$  and  $\xi' = \zeta_l$ . It is clear that  $\zeta \preceq \zeta_l$ . Now, according to (11) and (12), we can take  $\Delta'_\varrho = \{L_i - L_n, L_n - L_i\}$ . Hence, for  $i > 0$ ,

$$\mathfrak{g}_{2i}^{\zeta} \cap \mathfrak{m}_\varrho^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta'_\varrho \cap \Delta_{\zeta}^{2i}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}) \cap \mathfrak{m}_\varrho^{\mathbb{C}},$$

where  $\Delta_{\zeta}^{2i} = \{\alpha \in \Delta \mid \alpha(\zeta) = 2i\}$ . Since

$$(L_j - L_n)(\zeta) = (\alpha_j + \alpha_{j+1} + \dots + \alpha_{n-1})(\zeta) = 2|I_2 \cap \{j, \dots, n-1\}|i,$$

we have

$$\Delta'_\varrho \cap \Delta_{\zeta}^{2i} = \{L_j - L_n \mid 1 \leq j \leq l, \text{ and } |I_2 \cap \{j, \dots, l\}| = i\}.$$

Then, given a root  $\alpha = L_j - L_n \in \Delta'_\varrho \cap \Delta_{\zeta}^{2i}$  (in particular,  $j \leq l$ ) we have  $\alpha(\zeta - \zeta_l) = (2i - 1)i$ , which means that  $\mathfrak{g}_\alpha \subset \mathfrak{g}_{2i-1}^{\zeta - \zeta_l}$ . Consequently,

$$\mathfrak{g}_{2i}^{\zeta} \cap \mathfrak{m}_\varrho^{\mathbb{C}} \subset \bigoplus_{0 \leq j < 2i} \mathfrak{g}_j^{\zeta - \zeta_l}.$$

Since  $\mathfrak{g}_{2i-1}^{\zeta} = \{0\}$  for all  $i$ , we conclude that (14) holds, and the statement follows from Propositions 15 and 16.  $\square$

It is known [3] that there are no full harmonic maps  $\varphi : S^2 \rightarrow \mathbb{R}P^{2n-1}$ . The class of harmonic maps associated to  $(\zeta_l, \sigma_{\varrho, l})$  consists precisely of those  $\varphi$  with  $\varphi(S^2)$  contained, up to isometry, in some  $\mathbb{R}P^{2l}$ , as shown in the next theorem.

**Theorem 20.** Given  $1 \leq l \leq n - 1$ , any harmonic map  $\varphi : S^2 \rightarrow \mathbb{R}P^{2n-1}$  in the class  $(\zeta_l, \sigma_{\varrho, l})$  is given by

$$\varphi = R \cap (A \oplus \overline{A})^\perp, \quad (19)$$

where  $R$  is a constant  $2l + 1$ -dimensional subspace of  $\mathbb{R}^{2n}$  and  $A$  is a holomorphic isotropic subbundle of  $S^2 \times R$  of rank  $l$  satisfying  $\partial A \subseteq \overline{A}^\perp$ . The corresponding extended solutions have uniton number 2 with respect to the standard representation of  $SO(2n)$ .

*Proof.* Let  $\varphi : S^2 \rightarrow \mathbb{R}P^{2n-1}$  be a harmonic map in the class  $(\zeta_l, \sigma_{\varrho, l})$ . This means that  $\varphi$  admits an extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta_l}^{\sigma_{\varrho, l}}(SO(2n))$ . Up to isometry,  $\varphi$  is given by  $\Phi_{-1}$ , which takes values in  $P_{\zeta_l}^{\sigma_{\varrho, l}} = \exp(\pi\zeta_l)P_e^{\sigma_e}$ . This connected component is identified with  $\mathbb{R}P^{2n-1}$  via

$$g \cdot V_0 \mapsto \exp(\pi\zeta_l)g\sigma_{\varrho}(g^{-1}). \quad (20)$$

Write  $\gamma_{\zeta_l}(\lambda) = \lambda^{-1}\pi_{V_l} + \pi_{V_l \oplus \overline{V}_l}^\perp + \lambda\pi_{\overline{V}_l}$ , where  $V_l$  is the  $l$ -dimensional isotropic subspace spanned by  $\overline{u}_1, \dots, \overline{u}_l$ .

We have  $r(\zeta_l) = 2$  if  $l > 1$  and  $r(\zeta_1) = 1$ . Consequently, by Theorem 17,

$$(\mathfrak{u}_{\zeta_l}^0)_{\sigma_{\varrho, l}} = (\mathfrak{p}_0^{\zeta_l})^\perp \cap \mathfrak{k}_{\sigma_{\varrho, l}}^{\mathbb{C}} \oplus \lambda(\mathfrak{p}_1^{\zeta_l})^\perp \cap \mathfrak{m}_{\sigma_{\varrho, l}}^{\mathbb{C}}.$$

Here  $(\mathfrak{p}_1^{\zeta_l})^\perp = \mathfrak{g}_2^{\zeta_l}$ , which is the null space for  $l = 1$ . For  $l > 1$ , since  $\zeta_l = E_1 + \dots + E_l$ , we have  $\mathfrak{g}_2^{\zeta_l} = \{L_i + L_j \mid 1 \leq i < j \leq l\} \subset \Delta(\mathfrak{k}_\varrho)$  and, from (6),

$$\mathfrak{m}_{\sigma_{\varrho, l}}^{\mathbb{C}} = \bigoplus \mathfrak{g}_{2i+1}^{\zeta_l} \cap \mathfrak{k}_\varrho^{\mathbb{C}} \oplus \bigoplus \mathfrak{g}_{2i}^{\zeta_l} \cap \mathfrak{m}_\varrho^{\mathbb{C}}.$$

Hence  $(\mathfrak{p}_1^{\zeta_l})^\perp \cap \mathfrak{m}_{\sigma_{\varrho, l}}^{\mathbb{C}} = \mathfrak{g}_2^{\zeta_l} \cap \mathfrak{m}_\varrho^{\mathbb{C}} = \{0\}$ . Then, for any  $l \geq 1$ , we can write  $\Phi = \exp C \cdot \gamma_{\zeta_l}$  for some holomorphic function

$$C : S^2 \setminus D \rightarrow (\mathfrak{p}_0^{\zeta_l})^\perp \cap \mathfrak{k}_{\sigma_{\varrho, l}}^{\mathbb{C}} = (\mathfrak{g}_1^{\zeta_l} \oplus \mathfrak{g}_2^{\zeta_l}) \cap \mathfrak{k}_{\sigma_{\varrho, l}}^{\mathbb{C}},$$

which means that  $\Phi$  is a  $S^1$ -invariant extended solution with uniton number 2:

$$\Phi_\lambda = \lambda^{-1}\pi_W + \pi_{W \oplus \overline{W}}^\perp + \lambda\pi_{\overline{W}}, \quad (21)$$

where  $W$  is a holomorphic isotropic subbundle of  $S^2 \times \mathbb{R}^{2n}$  of rank  $l$  satisfying the superhorizontality condition  $\partial W \subseteq \overline{W}^\perp$ .

Set  $\tilde{V}_l = V_l \oplus \overline{V}_l$  and  $\tilde{W} = W \oplus \overline{W}$ . The  $T_{\sigma_{\varrho,l}}$ -invariance of  $\Phi$  implies that

$$[\pi_W, \pi_{V_0 \oplus \tilde{V}_l}] = 0. \quad (22)$$

Now, write  $\varphi = g \cdot V_0$  and consider the identification (20). We must have

$$\Phi_{-1} = \exp(\pi \zeta_l) g \sigma_{\varrho} (g^{-1}) = \exp(\pi \zeta_l) (\pi_{\varphi} - \pi_{\varphi}^\perp) s_0. \quad (23)$$

From (21) and (23) we obtain

$$\pi_{\varphi} - \pi_{\varphi}^\perp = \text{Ad}(s_0) (\pi_{V_0 \oplus \tilde{V}_l} \pi_{\tilde{W}}^\perp + \pi_{V_0 \oplus \tilde{V}_l}^\perp \pi_{\tilde{W}} - \pi_{V_0 \oplus \tilde{V}_l} \pi_{\tilde{W}} - \pi_{V_0 \oplus \tilde{V}_l}^\perp \pi_{\tilde{W}}^\perp). \quad (24)$$

In view of (22), we see that  $\pi_{V_0 \oplus \tilde{V}_l} \pi_{\tilde{W}}^\perp + \pi_{V_0 \oplus \tilde{V}_l}^\perp \pi_{\tilde{W}}$  is an orthogonal projection, and (24) implies that this must be an orthogonal projection onto a 1-dimensional real subspace. Then, one of its two terms vanishes, that is either  $\tilde{W} \subset V_0 \oplus \tilde{V}_l$  or  $\tilde{W}^\perp \subset (V_0 \oplus \tilde{V}_l)^\perp$ . For dimensional reasons, we see that the second case can not occur. Hence, we have

$$\pi_{\varphi} = \text{Ad}(s_0) (\pi_{V_0 \oplus \tilde{V}_l} \pi_{\tilde{W}}^\perp) = \pi_{V_0 \oplus \tilde{V}_l} \text{Ad}(s_0) (\pi_{\tilde{W}}^\perp),$$

that is (19) holds with  $R = V_0 \oplus V_l \oplus \overline{V}_l$  and  $A = s_0(W)$ . □

**Remark 4.** If  $\varphi$  is full in  $R$ , then the isotropic subbundle  $A$  is the  $l$ -osculating space of some full totally isotropic holomorphic map  $f$  from  $S^2$  into the complex projective space of  $R$ , the so called *directrix curve* of  $\varphi$ . That is, in a local system of coordinates  $(U, z)$ , we have  $A(z) = \text{Span}\{g, g', \dots, g^{(l-1)}\}$ , where  $g$  is a lift of  $f$  over  $U$  and  $g^{(r)}$  the  $r$ -th derivative of  $g$  with respect to  $z$ . Hence, formula (19) agrees with the classification given in Corollary 6.11 of [9].

**Example 1.** Let us consider the case  $n = 2$ . We have only one class of harmonic maps:  $(\zeta_1, \sigma_{\varrho,1})$ . From Theorem 20, any such harmonic map  $\varphi : S^2 \rightarrow \mathbb{R}P^3$  is given by  $\varphi = R \cap (A \oplus \overline{A})^\perp$ , where  $R$  is a constant 3-dimensional subspace of  $\mathbb{R}^4$  and  $A$  a holomorphic isotropic subbundle of  $S^2 \times R$  of rank 1 such that  $\partial A \subseteq \overline{A}^\perp$ . Taking into account Theorem 17, any such holomorphic subbundles  $A$  can be obtained from a meromorphic function  $a$  on  $S^2$  as follows.

We have  $\zeta_1 = E_1$  and the corresponding extended solutions have uniton number  $r(\zeta_1) = 1$  (with respect to the standard representation). Any extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta_1}^{\sigma_{\varrho,1}}(SO(4))$  is given by  $\Phi = \exp C \cdot \gamma_{\zeta_1}$ , with  $\gamma_{\zeta_1}(\lambda) = \lambda^{-1} \pi_{V_1} + \pi_{V_1 \oplus \overline{V}_1}^\perp + \lambda \pi_{\overline{V}_1}$ , for some holomorphic vector-valued function  $C : S^2 \setminus D \rightarrow (\mathfrak{u}_{\zeta_1}^0)_{\sigma_{\varrho,1}}$ , where

$$(\mathfrak{u}_{\zeta_1}^0)_{\sigma_{\varrho,1}} = (\mathfrak{p}_0^{\zeta_1})^\perp \cap \mathfrak{k}_{\sigma_{\varrho,1}}^\mathbb{C} = \mathfrak{g}_1^{\zeta_1} \cap \mathfrak{k}_{\sigma_{\varrho,1}}^\mathbb{C} = (\mathfrak{g}_{L_1-L_2} \oplus \mathfrak{g}_{L_1+L_2}) \cap \mathfrak{k}_{\sigma_{\varrho,1}}^\mathbb{C}.$$

Considering the root vectors  $X_{i,j}, Y_{i,j}, Z_{i,j}$  as defined in (17), we have  $Y_{1,2} = \sigma_{\varrho,1}(X_{1,2})$ . Hence  $C = a(z)(X_{1,2} + Y_{1,2})$  where  $a(z)$  is a meromorphic function on  $S^2$ . In this case, from (2), it follows that  $(\exp C)^{-1}(\exp C)_z = C_z$ , and it is clear that the extended solution condition for  $\Phi$  holds independently of the choice of the meromorphic function  $a(z)$ . Then, with respect to the complex basis  $\mathbf{u} = \{u_1, u_2, \overline{u}_1, \overline{u}_2\}$ ,

$$\exp C \cdot \gamma_{\zeta_1} = \begin{bmatrix} 1 & a & -a^2 & a \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{bmatrix} \cdot \gamma_{\zeta_1} \quad (25)$$

and the subbundle  $A$  of  $R = \text{Span}\{u_1, \overline{u}_1, u_2 + \overline{u}_2\}$  is given by  $A = \exp C \cdot V_1 = \text{span}\{(a^2, a, -1, a)\}$ , which satisfies  $\partial A \subseteq \overline{A}^\perp$ .

**Example 2.** Any harmonic two-sphere into  $\mathbb{R}P^5$  in the class  $(\zeta_1, \sigma_{\varrho,1})$  takes values in some  $\mathbb{R}P^3$  inside  $\mathbb{R}P^5$  and so it is essentially of the form (25). Next we consider the Weierstrass representation of harmonic spheres into  $\mathbb{R}P^5$  in the class  $(\zeta_2, \sigma_{\varrho,2})$ , which are given by  $\varphi = R \cap (A \oplus \overline{A})^\perp$ , where  $R$  is a constant 5-dimensional subspace of  $\mathbb{R}^6$  and  $A$  a holomorphic isotropic subbundle of  $S^2 \times R$  of rank 2 such that  $\partial A \subseteq \overline{A}^\perp$ . We have  $\zeta_2 = E_1 + E_2$ , then  $r(\zeta_2) = 2$ . Any extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta_2}^{\sigma_{\varrho,2}}(SO(6))$  is given by  $\Phi = \exp C \cdot \gamma_{\zeta_2}$ , with  $\gamma_{\zeta_2}(\lambda) = \lambda^{-1}\pi_{V_2} + \pi_{V_2 \oplus \overline{V}_2}^\perp + \lambda\pi_{\overline{V}_2}$ , for some holomorphic vector-valued function  $C : S^2 \setminus D \rightarrow (\mathfrak{u}_{\zeta_2}^0)_{\sigma_{\varrho,2}}$ , where

$$(\mathfrak{u}_{\zeta_2}^0)_{\sigma_{\varrho,2}} = \left( (\mathfrak{g}_{L_1-L_3} \oplus \mathfrak{g}_{L_1+L_3}) \cap \mathfrak{k}_{\sigma_{\varrho,2}}^\mathbb{C} \right) \oplus \left( (\mathfrak{g}_{L_2-L_3} \oplus \mathfrak{g}_{L_2+L_3}) \cap \mathfrak{k}_{\sigma_{\varrho,2}}^\mathbb{C} \right) \oplus \mathfrak{g}_{L_1+L_2}.$$

We have  $Y_{1,3} = \sigma_{\varrho,2}(X_{1,3})$  and  $Y_{2,3} = \sigma_{\varrho,2}(X_{2,3})$ . Hence we can write

$$C = a(z)(X_{1,3} + Y_{1,3}) + b(z)(X_{2,3} + Y_{2,3}) + c(z)Y_{1,2}$$

where  $a(z)$ ,  $b(z)$  and  $c(z)$  are meromorphic functions on  $S^2$ .

Now,  $\Phi = \exp C \cdot \gamma_{\zeta_2}$  is an extended solution if and only if, in the expression  $C_z - \frac{1}{2i}(\text{ad} C)C_z$ , which does not depend on  $\lambda$ , the component on  $\mathfrak{g}_2^{\zeta_2} = \mathfrak{g}_{L_1+L_2}$  vanishes. Since  $Y_{1,2} = [Y_{2,3}, X_{1,3}] = [X_{2,3}, Y_{1,3}]$  and  $[X_{1,3}, X_{2,3}] = [Y_{1,3}, Y_{2,3}] = 0$ , this holds if and only if  $c' = ba' - ab'$ , where prime denotes  $z$ -derivative. Since  $A = \exp C \cdot V_2$ , we can compute  $\exp C$  in order to conclude that the holomorphic subbundle  $A$  of  $R = \text{Span}\{u_1, u_2, \overline{u}_1, \overline{u}_2, u_3 + \overline{u}_3\}$  is given by

$$A = \text{Span}\{(a^2, ab + c, a, -1, 0, a), (ab - c, b^2, b, 0, -1, b)\}.$$

**5.1.2. Harmonic maps into Real Grassmannians.** Let  $\zeta'$  be a  $\varrho$ -semi-canonical element of  $SO(2n)$  given by (18), for some disjoint subsets  $I_1$  and  $I_2$  of  $\{1, \dots, n-1\}$ . By Proposition 9, we know that  $P_{\zeta'}^{\sigma_{\varrho}} \cong \mathbb{R}P^{2n-1}$  if and only if either  $I_1 = \emptyset$  or  $I_1 = \{n-1\}$ . More generally we have:

**Proposition 21.** If  $I_1 = \{i_1 > i_2 > \dots > i_r\}$  and  $d = \sum_{j=1}^r (-1)^{j+1} i_j$ , then  $P_{\zeta'}^{\sigma_{\varrho}} \cong G_{2d+1}(\mathbb{R}^{2n})$ .

*Proof.* For  $\zeta'$  of the form (18), set  $\zeta'_{I_1} = \sum_{i \in I_1} \zeta_i$ . Clearly,  $\exp \pi \zeta' = \exp \pi \zeta'_{I_1}$ , and, by Proposition 6,  $P_{\zeta'}^{\sigma_{\varrho}}$  is a symmetric space with involution

$$\tau = \text{Ad}(\exp \pi \zeta'_{I_1}) \circ \sigma_{\varrho} = \text{Ad}(s_0 \exp \pi \zeta'_{I_1}).$$

We have

$$\zeta'_{I_1} = r(E_1 + \dots + E_{i_r}) + (r-1)(E_{i_r+1} + \dots + E_{i_{r-1}}) + \dots + (E_{i_2+1} + \dots + E_{i_1}),$$

and consequently, with the convention  $V_{i_0} = V_n$  and  $V_{i_{r+1}} = \{0\}$ ,

$$\exp \pi \zeta'_{I_1} = \sum_{j=0}^r (-1)^j \pi_{i_j - i_{j+1}} + \sum_{j=0}^r (-1)^j \overline{\pi}_{i_j - i_{j+1}},$$

where  $\pi_{i_j - i_{j+1}}$  is the orthogonal projection onto  $V_{i_j} \cap V_{i_{j+1}}^\perp$  and  $\overline{\pi}_{i_j - i_{j+1}}$  the orthogonal projection onto the corresponding conjugate space. Hence, the  $+1$ -eigenspace of  $s_0 \exp \pi \zeta'_{I_1}$  has dimension  $2d+1$ , with  $d = \sum_{j=1}^r (-1)^{j+1} i_j$ , which means that  $P_{\zeta'}^{\sigma_{\varrho}} \cong G_{2d+1}(\mathbb{R}^{2n})$ .  $\square$

In particular, we have  $P_{\zeta_d}^{\sigma_{\varrho}} \cong G_{2d+1}(\mathbb{R}^{2n})$  for each  $d \in \{1, \dots, n-1\}$ .

**Theorem 22.** Each harmonic map from  $S^2$  into the real Grassmannian  $G_{2d+1}(\mathbb{R}^{2n})$  belongs to one of the following classes:  $(\zeta, \text{Ad} \exp \pi(\tilde{\zeta} - \zeta) \circ \sigma_{\varrho,l})$ , where  $\zeta$  and  $\tilde{\zeta}$  are  $\varrho$ -canonical elements such that  $\tilde{\zeta} \preceq \zeta$  and  $\tilde{\zeta} = \sum_{i \in I_1} \zeta_i + \zeta_l$ , where

- a)  $I_1 = \{i_1 > i_2 > \dots > i_r\}$  satisfies  $d = \sum_{j=1}^r (-1)^{j+1} i_j$ ;
- b)  $l \in \{0, 1, \dots, n-1\}$  and  $l \notin I_1$  (if  $l = 0$ , we set  $\zeta_0 = 0$ ).



*Proof.* We consider harmonic maps into  $P_{\zeta_d}^{\sigma_e} \cong G_{2d+1}(\mathbb{R}^{2n})$ . Let  $\zeta'$  be a  $\varrho$ -semi-canonical element and write  $\zeta' = \sum_{i \in I_1} \zeta_i + \sum_{i \in I_2} 2\zeta_i$  for some disjoint subsets  $I_1$  and  $I_2$  of  $\{1, \dots, n-1\}$ . By Proposition 21,  $P_{\zeta'}^{\sigma_e} \cong G_{2d+1}(\mathbb{R}^{2n})$  if and only if either  $d = \sum_{j=1}^r (-1)^{j+1} i_j$  or  $n-d-1 = \sum_{j=1}^r (-1)^{j+1} i_j$ , since  $G_{2d+1}(\mathbb{R}^{2n})$  and  $G_{2d'+1}(\mathbb{R}^{2n})$ , with  $d' = n-d-1$ , can be identified via  $V \mapsto V^\perp$ . However, it follows from the same reasoning as in the proof of Theorem 19 that, in the second case,  $P_{\zeta'}^{\sigma_e}$  does not coincide with the connected component  $P_{\zeta_d}^{\sigma_e}$ . So we only consider the  $\varrho$ -semi-canonical elements  $\zeta'$  with  $d = \sum_{j=1}^r (-1)^{j+1} i_j$ .

Set  $l = \max I_2$ . Next we check that the pair  $\zeta' \preceq \tilde{\zeta} = \sum_{i \in I_1} \zeta_i + \zeta_l$  satisfies the conditions of Proposition 16. Considering the same notations we used in the proof of Theorem 19, for each  $i > 0$  we have

$$\Delta'_e \cap \Delta_{\zeta'}^{2i} = \{L_j - L_n \mid 2|I_2 \cap \{j, \dots, l\}| + |I_1 \cap \{j, \dots, n-1\}| = 2i\}.$$

In particular, for  $i > 0$  and  $\alpha = L_j - L_n \in \Delta'_e \cap \Delta_{\zeta'}^{2i}$ , it is clear that  $\alpha(\zeta' - \tilde{\zeta})/i \leq 2i-1$ , and consequently

$$\mathfrak{g}_{2i}^{\zeta'} \cap \mathfrak{m}_e^{\mathbb{C}} \subset \bigoplus_{0 \leq j < 2i} \mathfrak{g}_j^{\zeta' - \tilde{\zeta}}.$$

For  $i > 0$ , we have the decomposition

$$\mathfrak{g}_{2i-1}^{\zeta'} \cap \mathfrak{k}_e^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta(\mathfrak{k}_e) \cap \Delta_{\zeta'}^{2i-1}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta'_e \cap \Delta_{\zeta'}^{2i-1}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\varrho(\alpha)}) \cap \mathfrak{k}_e^{\mathbb{C}}.$$

Given  $\alpha \in \mathfrak{g}_{2i-1}^{\zeta'}$ , since  $\alpha(\zeta')/i$  is odd, we must have  $\alpha(\zeta_j) \neq 0$  for some  $j \in I_1$ . Hence  $\alpha(\zeta' - \tilde{\zeta})/i < \alpha(\zeta')/i$  and we conclude that

$$\mathfrak{g}_{2i-1}^{\zeta'} \cap \mathfrak{k}_e^{\mathbb{C}} \subset \bigoplus_{0 \leq j < 2i-1} \mathfrak{g}_j^{\zeta' - \tilde{\zeta}}.$$

The statement of the theorem follows now from Propositions 15 and 16.  $\square$

Next we will study in detail the case  $G_3(\mathbb{R}^6)$ . Take as base point of  $G_3(\mathbb{R}^6)$  the 3-dimensional real subspace  $V_0 \oplus V_1 \oplus \overline{V}_1$ , where  $V_1$  is the one-dimensional isotropic subspace spanned by  $\overline{u}_1$ . This choice establishes the identification

$$G_3(\mathbb{R}^6) \cong SO(6)/SO(3) \times O(3)$$

and the corresponding involution is  $\sigma_{e,1} = \text{Ad}(\exp \pi \zeta_1) \circ \sigma_e$ . Following our classification procedure, we also identify  $G_3(\mathbb{R}^6)$  with  $P_{\zeta_1}^{\sigma_e}$  via the totally geodesic embedding (13). From Theorem 22, we have six classes of harmonic maps into  $G_3(\mathbb{R}^6)$ :

$$(\zeta_1, \sigma_e), (\zeta_1 + \zeta_2, \sigma_e), (\zeta_2, \sigma_{e,1}), (\zeta_1, \sigma_{e,2}), (\zeta_1 + \zeta_2, \sigma_{e,2}), (\zeta_2, \text{Ad}(\exp \pi \zeta_2) \circ \sigma_{e,1}).$$

**Theorem 23.** Let  $\varphi : S^2 \rightarrow G_3(\mathbb{R}^6)$  be an harmonic map.

- (1) If  $\varphi$  is associated to the pair  $(\zeta_1, \sigma_e)$  then  $\varphi$  is  $S^1$ -invariant and, up to isometry, is given by

$$\varphi = V_0 \oplus V \oplus \overline{V}, \tag{26}$$

where  $V$  is a holomorphic isotropic subbundle of  $S^2 \times V_0^\perp$  of rank 1 satisfying  $\partial V \subseteq \overline{V}^\perp$ .

- (2) If  $\varphi$  is associated to the pair  $(\zeta_1 + \zeta_2, \sigma_e)$  and is  $S^1$ -invariant, then, up to isometry,

$$\varphi = V_0 \oplus (W \cap V^\perp) \oplus (\overline{W \cap V^\perp}), \tag{27}$$

where  $V \subset W$  are holomorphic isotropic subbundles of  $S^2 \times V_0^\perp$  of rank 1 and 2, respectively, satisfying  $\partial V \subset W$  and  $\partial W \subset \overline{W}^\perp$ .

- (3) If  $\varphi$  is associated to the pair  $(\zeta_2, \sigma_{e,1})$  and is  $S^1$ -invariant, then, up to isometry,

$$\varphi = \{(L_1 \oplus \overline{L}_1)^\perp \cap (V_0 \oplus V_1 \oplus \overline{V}_1)\} \oplus (L_2 \oplus \overline{L}_2), \tag{28}$$

where  $L_1$  and  $L_2$  are holomorphic isotropic bundle lines of  $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)$  and  $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)^\perp$ , respectively.

The corresponding extended solutions have uniton number 2, 4, and 2, respectively, with respect to the standard representation of  $SO(6)$ . The harmonic maps in the classes  $(\zeta_1, \sigma_{\varrho,2})$ ,  $(\zeta_1 + \zeta_2, \sigma_{\varrho,2})$ , and  $(\zeta_2, \text{Ad}(\exp \pi \zeta_2) \circ \sigma_{\varrho,1})$  are precisely the orthogonal complements of the harmonic maps in the classes  $(\zeta_1, \sigma_{\varrho})$ ,  $(\zeta_1 + \zeta_2, \sigma_{\varrho})$ , and  $(\zeta_2, \sigma_{\varrho,1})$ , respectively.

*Proof.* For the first two classes, and according to our classification procedure, we identify  $G_3(\mathbb{R}^6)$  with  $P_{\zeta_1}^{\sigma_{\varrho}}$  via the totally geodesic embedding  $g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1) \mapsto \exp(\pi \zeta_1) g \sigma_{\varrho,1}(g^{-1})$ . In these two cases,  $T_{\sigma_{\varrho}}$ -invariant extended solutions  $\Phi$  associated to harmonic maps  $\varphi = g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1)$  satisfy

$$\Phi_{-1} = \exp(\pi \zeta_1) g \sigma_{\varrho,1}(g^{-1}) = \exp(\pi \zeta_1) (\pi_{\varphi} - \pi_{\varphi}^{\perp}) \exp(\pi \zeta_1) s_0. \quad (29)$$

First we consider the harmonic maps associated to the pair  $(\zeta_1, \sigma_{\varrho})$ . We have  $r(\zeta_1) = 1$  and

$$(\mathfrak{u}_{\zeta_1}^0)_{\sigma_{\varrho}} = (\mathfrak{p}_0^{\zeta_1})^{\perp} \cap \mathfrak{k}_{\varrho}^{\mathbb{C}} = \mathfrak{g}_1^{\zeta_1} \cap \mathfrak{k}_{\varrho}^{\mathbb{C}}.$$

Consequently any such harmonic map is  $S^1$ -invariant. Write  $\gamma_{\zeta_1}(\lambda) = \lambda^{-1} \pi_{V_1} + \pi_{V_1 \oplus \overline{V}_1}^{\perp} + \lambda \pi_{\overline{V}_1}$ , where  $V_1$  is the one-dimensional isotropic space spanned by  $\overline{u}_1$ . Let  $\Phi : S^2 \setminus D \rightarrow U_{\zeta_1}^{\sigma_{\varrho}}$  be an extended solution associated to the harmonic map  $\varphi$ . Then, by  $S^1$ -invariance, we can write

$$\Phi_{\lambda} = \lambda^{-1} \pi_V + \pi_{V \oplus \overline{V}}^{\perp} + \lambda \pi_{\overline{V}}, \quad (30)$$

where  $V$  is a holomorphic isotropic subbundle of  $S^2 \times \mathbb{R}^6$  of rank 1 satisfying  $\partial V \subseteq \overline{V}^{\perp}$ . The  $T_{\sigma_{\varrho}}$ -invariance of  $\Phi$  implies that  $V_0 \subset (V \oplus \overline{V})^{\perp}$ . Equating (29) and (30), we get, up to isometry,  $\varphi = V_0 \oplus V \oplus \overline{V}$ .

For the case  $(\zeta_1 + \zeta_2, \sigma_{\varrho})$ , since

$$\gamma_{\zeta_1 + \zeta_2}(\lambda) = \lambda^{-2} \pi_{V_1} + \lambda^{-1} \pi_{V_2 \cap V_1^{\perp}} + \pi_{V_2 \oplus \overline{V}_2}^{\perp} + \lambda \pi_{\overline{V}_2 \cap \overline{V}_1^{\perp}} + \lambda^2 \pi_{\overline{V}_1}, \quad (31)$$

any  $S^1$ -invariant harmonic map  $\varphi$  in this class admits an extended solution of the form

$$\Phi_{\lambda} = \lambda^{-2} \pi_V + \lambda^{-1} \pi_{W \cap V^{\perp}} + \pi_{W \oplus \overline{W}}^{\perp} + \lambda \pi_{\overline{W} \cap \overline{V}^{\perp}} + \lambda^2 \pi_{\overline{V}}, \quad (32)$$

where  $V \subset W$  are holomorphic isotropic subbundles of rank 1 and 2, respectively, satisfying  $\partial V \subset W$  and  $\partial W \subset \overline{W}^{\perp}$ . By  $T_{\sigma_{\varrho}}$ -invariance, we must have  $V_0 \subset (W \oplus \overline{W})^{\perp}$ , hence  $V \subset W$  are subbundles of  $S^2 \times V_0^{\perp}$ . Equating (29) and (32), we get (27).

For the case  $(\zeta_2, \sigma_{\varrho,1})$ , we identify  $G_3(\mathbb{R}^6)$  with  $P_{\zeta_2}^{\sigma_{\varrho,1}} = \exp \pi \zeta_1 P_{\zeta_2 - \zeta_1}^{\sigma_{\varrho}}$  via the totally geodesic embedding

$$g \cdot (V_0 \oplus V_1 \oplus \overline{V}_1) \mapsto g \sigma_{\varrho,1}(g^{-1}). \quad (33)$$

Extended solutions  $\Phi$  associated to  $S^1$ -invariant harmonic maps in this class must be of the form

$$\Phi_{\lambda} = \lambda^{-1} \pi_W + \pi_{W \oplus \overline{W}}^{\perp} + \lambda \pi_{\overline{W}}, \quad (34)$$

where  $W$  is a holomorphic isotropic subbundle of rank 2. By  $T_{\sigma_{\varrho,1}}$ -invariance, we must have  $[\pi_W, \pi_{V_0 \oplus V_1 \oplus \overline{V}_1}] = 0$ , which means that  $W$  must be of the form  $W = L_1 \oplus L_2$ , where  $L_1$  and  $L_2$ , respectively, are holomorphic isotropic bundle lines of  $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)$  and  $S^2 \times (V_0 \oplus V_1 \oplus \overline{V}_1)^{\perp}$ .

On the other hand, in view of (33), we have  $\Phi_{-1} = (\pi_{\varphi} - \pi_{\varphi}^{\perp}) \exp(\pi \zeta_1) s_0$ . Equating this with (34), we conclude that (28) holds. The remaining cases are treated similarly.  $\square$

**Remark 5.** The first two classes of  $S^1$ -invariant harmonic maps  $\varphi : S^2 \rightarrow G_3(\mathbb{R}^6)$  in Theorem 23 factor through  $G_2(\mathbb{R}^5)$ . That is, for any such harmonic map  $\varphi$ , there exists  $\tilde{\varphi} : S^2 \rightarrow G_2(\mathbb{R}^5)$ , where we identify  $\mathbb{R}^5$  with  $V_0^{\perp}$ , such that  $\varphi = V_0 \oplus \tilde{\varphi}$ . An explicit construction of all harmonic maps from  $S^2$  into  $G_2(\mathbb{R}^n)$  can be found in [16]. In that paper, harmonic maps of the form (26) are called *real mixed pairs*. We emphasise that the harmonic maps into  $G_3(\mathbb{R}^6)$  associated to extended solutions in the corresponding unstable manifolds need not to factor through  $G_2(\mathbb{R}^5)$  in the same way.

Let us consider the case  $(\zeta_1 + \zeta_2, \sigma_\varrho)$ . Taking into account the Weierstrass representation of Theorem 17, any extended solution  $\Phi : S^2 \setminus D \rightarrow U_\zeta^{\sigma_\varrho}(SO(6))$ , with  $\zeta = \zeta_1 + \zeta_2$ , can be written as  $\Phi = \exp C \cdot \gamma_\zeta$ , for some meromorphic vector-valued function  $C : S^2 \rightarrow (\mathfrak{u}_\zeta^0)_{\sigma_\varrho}$ . We have  $r(\zeta) = 3$  and

$$(\mathfrak{u}_\zeta^0)_{\sigma_\varrho} = (\mathfrak{g}_1^\zeta \oplus \mathfrak{g}_2^\zeta \oplus \mathfrak{g}_3^\zeta) \cap \mathfrak{k}_\varrho^\mathbb{C} \oplus \lambda(\mathfrak{g}_2^\zeta \oplus \mathfrak{g}_3^\zeta) \cap \mathfrak{m}_\varrho^\mathbb{C} \oplus \lambda^2 \mathfrak{g}_3^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C}.$$

Moreover,

$$\begin{aligned} \mathfrak{g}_1^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C} &= \mathfrak{g}_{L_1-L_2} \oplus \{(\mathfrak{g}_{L_2-L_3} \oplus \mathfrak{g}_{L_2+L_3}) \cap \mathfrak{k}_\varrho^\mathbb{C}\}, & \mathfrak{g}_2^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C} &= (\mathfrak{g}_{L_1+L_3} \oplus \mathfrak{g}_{L_1-L_3}) \cap \mathfrak{k}_\varrho^\mathbb{C}, \\ \mathfrak{g}_3^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C} &= \mathfrak{g}_{L_1+L_2}, & (\mathfrak{g}_2^\zeta \oplus \mathfrak{g}_3^\zeta) \cap \mathfrak{m}_\varrho^\mathbb{C} &= \mathfrak{g}_2^\zeta \cap \mathfrak{m}_\varrho^\mathbb{C} = (\mathfrak{g}_{L_1-L_3} \oplus \mathfrak{g}_{L_1+L_3}) \cap \mathfrak{m}_\varrho^\mathbb{C}. \end{aligned}$$

Write

$$C = C_0 + \lambda C_1 + \lambda^2 C_2, \quad C_0 = c_0^1 + c_0^2 + c_0^3, \quad C_1 = c_1^2 + c_1^3, \quad C_2 = c_2^3 \quad (35)$$

where the functions  $c_0^i : S^2 \rightarrow \mathfrak{g}_i^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C}$ ,  $c_1^i : S^2 \rightarrow \mathfrak{g}_i^\zeta \cap \mathfrak{m}_\varrho^\mathbb{C}$ , and  $c_2^3 : S^2 \rightarrow \mathfrak{g}_3^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C}$  are meromorphic functions. Clearly,  $c_1^3 = 0$ . Consider the root vectors defined by (17). Since  $\sigma_\varrho(X_{2,3}) = -Y_{2,3}$  and  $\sigma_\varrho(X_{1,3}) = -Y_{1,3}$ , we can write

$$c_0^1 = aX_{1,2} + b(X_{2,3} - Y_{2,3}), \quad c_0^2 = c(X_{1,3} - Y_{1,3}), \quad c_0^3 = dY_{1,2}, \quad c_1^2 = e(X_{1,3} + Y_{1,3}), \quad c_2^3 = fX_{1,2}$$

in terms of  $\mathbb{C}$ -valued meromorphic functions  $a, b, c, d, e, f$ .

Taking into account the results of Section 3.2,  $\Phi = \exp C \cdot \gamma_\zeta$  is an extended solution if and only if, in the expression

$$(\exp C)^{-1}(\exp C)_z = C_z - \frac{1}{2!}(\text{ad} C)C_z + \frac{1}{3!}(\text{ad} C)^2 C_z,$$

we have:

a) the independent coefficient should have zero component in each  $\mathfrak{g}_2^\zeta$  and  $\mathfrak{g}_3^\zeta$ , that is

$$c_{0z}^2 - \frac{1}{2}[c_0^1, c_{0z}^1] = 0, \quad c_{0z}^3 - \frac{1}{2}[c_0^1, c_{0z}^2] - \frac{1}{2}[c_0^2, c_{0z}^1] + \frac{1}{6}[c_0^1, [c_0^1, c_{0z}^1]] = 0; \quad (36)$$

b) the  $\lambda$  coefficient should have zero component in  $\mathfrak{g}_3^\zeta$ , that is

$$[c_0^1, c_{1z}^2] + [c_1^2, c_{0z}^1] = 0. \quad (37)$$

From equations (36) we get the equations (prime denotes  $z$ -derivative)

$$2c' = ab' - ba', \quad 3d' = 3cb' - bc'; \quad (38)$$

on the other hand, observe that (37) always holds since

$$[c_0^1, c_{1z}^2] + [c_1^2, c_{0z}^1] \in [\mathfrak{g}_1^\zeta \cap \mathfrak{k}_\varrho^\mathbb{C}, \mathfrak{g}_2^\zeta \cap \mathfrak{m}_\varrho^\mathbb{C}] \subset \mathfrak{g}_3^\zeta \cap \mathfrak{m}_\varrho^\mathbb{C} = \{0\}.$$

Hence we conclude that, any extended solution  $\Phi : S^2 \setminus D \rightarrow U_\zeta^{\sigma_\varrho}(SO(6))$ , with  $\zeta = \zeta_1 + \zeta_2$ , of the form  $\Phi = \exp C \cdot \gamma_\zeta$ , can be constructed as follows: choose arbitrary meromorphic functions  $a, b, e$  and  $f$ ; integrate equations (38) to obtain the meromorphic functions  $c$  and  $d$ ;  $C$  is then given by (35).

**Example 3.** Choose  $a(z) = b(z) = z$ . From (38), we can take  $c(z) = 1$  and  $d(z) = z$ . This data defines the matrix  $C_0$  and the  $S^1$ -invariant extended solution  $\Phi^0 = \exp C_0 \cdot \gamma_\zeta$ , where the loop  $\gamma_\zeta$ , with  $\zeta = \zeta_1 + \zeta_2$ , is given by (31). The extended solutions  $\Phi : S^2 \rightarrow U_\zeta^{\sigma_\varrho}(SO(6))$  satisfying  $\Phi^0 = u_\zeta \circ \Phi$  are of the form  $\Phi = \exp C \cdot \gamma_\zeta$ , where the matrix  $C = C_0 + C_1\lambda + C_2\lambda^2$  is given by

$$C = \begin{pmatrix} 0 & z & 1 & 0 & z & -1 \\ 0 & 0 & z & -z & 0 & -z \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 0 & 0 \\ 0 & 0 & 0 & -1 & -z & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & e\lambda & 0 & f\lambda^2 & -e\lambda \\ 0 & 0 & 0 & -f\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & e\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e\lambda & 0 & 0 \end{pmatrix},$$

with respect to the complex orthonormal basis  $\mathbf{u} = \{u_1, u_2, u_3, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$ , where  $e$  and  $f$  are arbitrary meromorphic functions on  $S^2$ . The holomorphic vector bundles  $V$  and  $W$  associated to the  $S^1$ -invariant extended solution  $\exp C_0 \cdot \gamma_\zeta$  are given by  $V = \exp C_0 \cdot V_1$  and  $W = \exp C_0 \cdot V_2$ , and we have, with respect to the basis  $\mathbf{u}$ ,

$$V = \text{span}\{(12 - 12z^2 - z^4, -4z^3, 12 - 6z^2, 12, -12z, -12 + 6z^2)\}$$

$$W = \text{span}\{(6z + z^3, 3z^2, 3z, 0, 3, -3z)\} \oplus V.$$

**5.2. Outer symmetric  $SU(2n+1)$ -spaces.** Let  $E_j$  be the square  $(m \times m)$ -matrix whose  $(j, j)$ -entry is 1 and all other entries are 0. The complexification  $\mathfrak{t}^\mathbb{C}$  of the algebra  $\mathfrak{t}$  of diagonal matrices  $\sum a_i E_i$ , with  $a_i \in \mathbb{C}$  and  $\sum a_i = 0$ , is a Cartan subalgebra of  $\mathfrak{su}(m)^\mathbb{C}$ . Let  $\{L_1, \dots, L_m\}$  be the dual basis of  $\{E_1, \dots, E_m\}$ , that is  $L_i(E_j) = \text{id}_{ij}$ . The roots of  $\mathfrak{su}(m)$  are the vectors  $L_i - L_j$ , with  $i \neq j$  and  $1 \leq i, j \leq m-1$  and  $\Delta^+ = \{L_i - L_j\}_{i < j}$  is a positive root system with positive simple roots  $\alpha_i = L_i - L_{i+1}$ , for  $1 \leq i \leq m-1$ . For  $i \neq j$ , the matrix  $X_{i,j}$  whose  $(i, j)$  entry is 1 and all other entries are 0 generate the root space  $\mathfrak{g}_{L_i - L_j}$ . The dual basis of  $\Delta_0 = \{\alpha_1, \dots, \alpha_{m-1}\}$  in  $\mathfrak{t}^*$  is formed by the matrices

$$H_i = \frac{m-i}{m}(E_1 + \dots + E_i) - \frac{i}{m}(E_{i+1} + \dots + E_m).$$

**5.2.1. Special Lagrangian spaces.** Consider on  $\mathbb{R}^{2m}$  the standard inner product  $\langle \cdot, \cdot \rangle$  and the canonical orthonormal basis  $\mathbf{e}^{2m} = \{e_1, \dots, e_{2m}\}$ . Define the orthogonal complex structure  $I$  by  $I(e_i) = e_{2m+1-i}$ , for  $i \in \{1, \dots, m\}$ . A *Lagrangian subspace* of  $\mathbb{R}^{2m}$  (with respect to  $I$ ) is a  $m$ -dimensional subspace  $L$  such that  $IL \perp L$ . Let  $\mathcal{L}_m$  be the space of all Lagrangian subspaces of  $\mathbb{R}^{2m}$  and  $L_0 \in \mathcal{L}_m$  the Lagrangian subspace generated by  $\mathbf{e}^m = \{e_1, \dots, e_m\}$ . The unitary group  $U(m)$  acts transitively on  $\mathcal{L}_m$ , with isotropy group at  $L_0$  equal to  $SO(m)$ , and  $\mathcal{L}_m$  is a reducible symmetric space that can be identified with  $U(m)/SO(m)$  (see [18] for details).

The space  $\mathcal{L}_m$  can also be interpreted as the set of all orthogonal linear maps  $\tau : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$  satisfying  $\tau^2 = e$  and  $I\tau = -\tau I$ . Indeed, if  $V_\pm$  are the  $\pm 1$  eigenspaces of  $\tau$ , then  $IV_+ = V_-$  and  $IV_+ \perp V_+$ , that is  $V_+$  is Lagrangian. From this point of view,  $U(m)$  acts on  $\mathcal{L}_m$  by conjugation:  $g \cdot \tau = g\tau g^{-1}$ . Let  $\tau_0 \in \mathcal{L}_m$  be the orthogonal linear map corresponding to  $L_0$ , that is,  $\tau_0|_{L_0} = e$  and  $\tau_0|_{L_0^\perp} = -e$ . The corresponding involution on  $U(m)$  is given by  $\sigma(g) = \tau_0 g \tau_0$  and the Cartan embedding  $\iota : \mathcal{L}_m \hookrightarrow U(m)$  is given by  $\iota(\tau) = \tau \tau_0$ .

The totally geodesic submanifold  $\mathcal{L}_m^s := SU(m)/SO(m)$  of  $U(m)/SO(m)$  is also known as the *space of special Lagrangian subspaces of  $\mathbb{R}^{2m}$* . It is an irreducible outer symmetric  $SU(m)$ -space.

**5.2.2. Harmonic maps into  $\mathcal{L}_{2n+1}^s$ .** Take  $m = 2n+1$ . The non-trivial involution  $\varrho$  of the Dynkin diagram of  $\mathfrak{su}(2n+1)^\mathbb{C}$  is given by  $\varrho(\alpha_i) = \alpha_{2n+1-i}$ . In particular,  $\varrho$  does not fix any root in  $\Delta_0$  and there exists only one class of outer symmetric  $SU(2n+1)$ -spaces. The semi-fundamental basis  $\pi_{\mathbf{e}_e}(\Delta_0) = \{\beta_1, \dots, \beta_n\}$  is given by  $\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n+1-i})$  whereas the dual basis  $\{\zeta_1, \dots, \zeta_n\}$  is given by

$$\zeta_i = H_i + H_{2n+1-i} = E_1 + \dots + E_i - (E_{2n+2-i} + \dots + E_{2n+1}),$$

for  $1 \leq i \leq n$ . Since each  $\zeta_i$  belongs to the integer lattice  $\mathfrak{I}(SU(2n+1))$ , the  $\varrho$ -semi-canonical elements of  $SU(2n+1)$  are precisely the elements  $\zeta = \sum_{i=1}^n m_i \zeta_i$  with  $m_i \in \{0, 1, 2\}$ .

Let  $\mathbf{e}^{2n+1} = \{e_1, \dots, e_{2n+1}\}$  be the canonical orthonormal basis of  $\mathbb{R}^{2n+1}$ . Identify  $\mathbb{C}^{2n+1}$  with  $(\mathbb{R}^{4n+2}, I)$ , where  $I$  is defined as above. Set

$$v_j = \frac{1}{\sqrt{2}}(e_j + ie_{2n+2-j}),$$

for  $1 \leq j \leq n$ ,  $v_{n+1} = e_{n+1}$  and  $v_{2n+2-j} = \bar{v}_j$ . Take the matrices  $E_j$  with respect to the complex basis  $\mathbf{v} = \{v_1, \dots, v_{2n+1}\}$  of  $\mathbb{C}^{2n+1}$ . Hence  $\tau_0 E_j \tau_0 = -E_{2n+2-j}$  and the fundamental involution  $\sigma_\varrho$  is given by  $\sigma_\varrho(g) = \tau_0 g \tau_0$ . The fundamental outer symmetric  $SU(2n+1)$ -space is the space of special Lagrangian subspaces  $\mathcal{L}_{2n+1}^s = SU(2n+1)/SO(2n+1)$ , and this is the unique outer symmetric  $SU(2n+1)$ -space.

Next we consider in detail harmonic maps into  $\mathcal{L}_3^s$ . In this case we have two non-zero  $\varrho$ -semi-canonical elements,  $\zeta_1$  and  $2\zeta_1$ , and consequently two classes of harmonic maps,  $(\zeta_1, \sigma_\varrho)$  and  $(\zeta_1, \sigma_{\varrho,1})$ . Since  $\zeta_1 = E_1 - E_3$ ,

we have  $r(\zeta_1) = (L_1 - L_3)(\zeta_1)/i = 2$ . Let  $W_1$ ,  $W_2$  and  $W_3$  be the complex one-dimensional images of  $E_1$ ,  $E_2$  and  $E_3$ , respectively. Any extended solution

$$\Phi : S^2 \setminus D \rightarrow U_{\zeta_1}^{\sigma_e}(SU(2n+1))$$

is given by  $\Phi = \exp C \cdot \gamma_{\zeta_1}$ , with  $\gamma_{\zeta_1}(\lambda) = \lambda^{-1}\pi_{W_3} + \pi_{W_2} + \lambda\pi_{W_1}$ , for some holomorphic vector-valued function  $C : S^2 \setminus D \rightarrow (\mathfrak{u}_{\zeta_1}^0)_{\sigma_e}$ , where

$$(\mathfrak{u}_{\zeta_1}^0)_{\sigma_e} = (\mathfrak{p}_0^{\zeta_1})^\perp \cap \mathfrak{k}_e^{\mathbb{C}} + \lambda(\mathfrak{p}_1^{\zeta_1})^\perp \cap \mathfrak{m}_e^{\mathbb{C}}$$

and

$$(\mathfrak{p}_0^{\zeta_1})^\perp \cap \mathfrak{k}_e^{\mathbb{C}} = (\mathfrak{g}_{L_1-L_2} \oplus \mathfrak{g}_{L_2-L_3} \oplus \mathfrak{g}_{L_1-L_3}) \cap \mathfrak{k}_e^{\mathbb{C}}, \quad (\mathfrak{p}_1^{\zeta_1})^\perp \cap \mathfrak{m}_e^{\mathbb{C}} = \mathfrak{g}_{L_1-L_3} \cap \mathfrak{m}_e^{\mathbb{C}}.$$

Let  $X_{i,j}$  be the square matrix whose  $(i,j)$  entry is 1 and all the other entries are 0, with respect to the basis  $\mathbf{v}$ . The root space  $\mathfrak{g}_{L_i-L_j}$  is spanned by  $X_{i,j}$ . We have  $\sigma_e(X_{1,2}) = -X_{2,3}$  and  $\sigma_e(X_{1,3}) = -X_{1,3}$  (consequently,  $\mathfrak{g}_{L_1-L_3} \subset \mathfrak{m}_e^{\mathbb{C}}$ ). Hence we can write  $C = C_0 + C_1\lambda$ , with  $C_0 = a(X_{1,2} - X_{2,3})$  and  $C_1 = bX_{1,3}$ , for some meromorphic functions  $a, b$  on  $S^2$ . The harmonicity equations do not impose any condition on these meromorphic functions, hence any harmonic map  $\varphi : S^2 \rightarrow \mathcal{L}_3^s$  in the class  $(\zeta_1, \sigma_e)$  admits an extended solution of the form

$$\Phi = \exp \begin{pmatrix} 0 & a & b\lambda \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix} \cdot \gamma_{\zeta_1} = \begin{pmatrix} 1 & a & \frac{1}{2}(-a^2 + 2b\lambda) \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{\zeta_1}, \quad (39)$$

and  $\varphi$  is recovered by setting  $\varphi = \Phi_{-1}\tau_0$ . Similarly, one can see that the class of harmonic maps in  $(\zeta_1, \sigma_{e,1})$  admits an extended solution of the form

$$\Phi = \begin{pmatrix} 1 & a & \frac{1}{2}(a^2 + 2b\lambda) \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{\zeta_1}, \quad (40)$$

with no restrictions on the meromorphic functions  $a$  and  $b$ .

H. Ma established (cf. Theorem 4.1 of [13]) that harmonic maps  $\varphi : S^2 \rightarrow \mathcal{L}_3^s$  are essentially of two types: 1)  $\iota_\sigma \circ \varphi$  is a *Grassmannian solution* obtained from a full harmonic map  $f : S^2 \rightarrow \mathbb{R}P^2 \subset \mathbb{C}P^2$ , where  $\iota_\sigma$  is the Cartan embedding of  $\mathcal{L}_3^s$  in  $SU(3)$ ; 2) up to left multiplication by a constant,  $\iota_\sigma \circ \varphi$  is of the form  $(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$ , where  $\beta_1$  is a *Frenet pair* associated to a full totally isotropic holomorphic map  $g : S^2 \rightarrow \mathbb{C}P^2$  and  $\beta_2$  is a rank 1 holomorphic subbundle of  $G'(g)^\perp$ , where  $G'(g)$  is the first *Gauss bundle* of  $g$ . Observe that if, in the second case,  $\beta_2$  coincides with  $g$ , then  $\iota_\sigma \circ \varphi$  is a Grassmannian solution obtained from the full harmonic map  $f := G'(g)$  from  $S^2$  to  $\mathbb{R}P^2$ , that is,  $\varphi$  is of type 1). Comparing this with our description, it is not difficult to see that harmonic maps of type 1) are  $S^1$ -invariant extended solutions (take  $b = 0$  in (39) and (40)) and harmonic maps of type 2) are associated to extended solutions with values in the corresponding unstable manifolds (which corresponds to an arbitrary choice of  $b$  in (39) and (40)). H. Ma also established a purely algebraic explicit construction of such harmonic maps in terms of meromorphic data on  $S^2$ , which is consistent with our results.

**5.3. Outer symmetric  $SU(2n)$ -spaces.** With the same notations of Section 5.2, the non-trivial involution  $\varrho$  of the Dynkin diagram of  $\mathfrak{su}(2n)$  is given by  $\varrho(\alpha_i) = \alpha_{2n-i}$ , and  $\varrho$  fixes the root  $\alpha_n$ . The semi-fundamental basis  $\pi_{\mathfrak{k}_e}(\Delta_0) = \{\beta_1, \dots, \beta_{n-1}\}$  is given by  $\beta_1 = \alpha_n$  and  $\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i})$  if  $i \geq 2$ ; whereas its dual basis  $\{\zeta_1, \dots, \zeta_{n-1}\}$  is given by

$$\begin{aligned} \zeta_1 &= H_n = \frac{1}{2}(E_1 + \dots + E_n) - \frac{1}{2}(E_{n+1} + \dots + E_{2n}) \\ \zeta_i &= H_{i-1} + H_{2n-i+1} = E_1 + \dots + E_{i-1} - (E_{2n+2-i} + \dots + E_{2n}), \quad \text{for } 2 \leq i \leq n-1. \end{aligned}$$

By Theorem 7, there exist two conjugacy classes of outer involutions: the fundamental outer involution  $\sigma_\varrho$  and  $\sigma_{\varrho,1}$ . These outer involutions correspond to the symmetric spaces  $SU(2n)/Sp(n)$  and  $SU(2n)/SO(2n)$ , respectively. Observe that  $\zeta_1$  does not belong to the integer lattice  $\mathcal{J}'(SU(2n)^{\sigma_e})$  since  $\exp 2\pi\zeta_1 = -e$ .

**5.3.1. Harmonic maps into the space of special unitary quaternionic structures on  $\mathbb{C}^{2n}$ .** A unitary quaternionic structure on the standard hermitian space  $(\mathbb{C}^{2n}, \langle \cdot, \cdot \rangle)$  is a conjugate linear map  $J : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$  satisfying  $J^2 = -Id$  and  $\langle v, w \rangle = \langle Jw, Jv \rangle$  for all  $v, w \in \mathbb{C}^{2n}$ . Consider as base point the quaternionic structure  $J_o$  defined by  $J_o(e_i) = e_{2n+1-i}$  for each  $1 \leq i \leq n$ , where  $\mathbf{e}^{2n} = \{e_1, \dots, e_{2n}\}$  is the canonical hermitian basis of  $\mathbb{C}^{2n}$ . The unitary group  $U(2n)$  acts transitively on the space of unitary quaternionic structures on  $\mathbb{C}^{2n}$  with isotropy group at  $J_o$  equal to  $Sp(n)$ , and thus  $M = U(2n)/Sp(n)$ . This is a reducible symmetric space with involution  $\sigma : U(2n) \rightarrow U(2n)$  given by  $\sigma(X) = J_o X J_o^{-1}$ , but the totally geodesic submanifold  $\mathcal{Q}_n^s := SU(2n)/Sp(n)$  is an irreducible symmetric space, which we call the space of *special unitary quaternionic structures* on  $\mathbb{C}^{2n}$  (see [18] for details). If we consider the matrices  $E_i$  with respect to the complex basis  $\mathbf{v} = \{v_1, \dots, v_{2n}\}$  defined by

$$v_j = \frac{1}{\sqrt{2}}(e_j + ie_{2n+1-j}), \quad (41)$$

for  $1 \leq j \leq n$ , and  $v_{2n+1-j} = \bar{v}_j$ , we see that  $J_o E_j J_o^{-1} = -E_{2n+1-j}$ , and consequently we have  $\sigma = \sigma_\varrho$ .

Next we consider with detail harmonic maps into  $\mathcal{Q}_2^s$ .

**Proposition 24.** Each harmonic map  $\varphi : S^2 \rightarrow \mathcal{Q}_2^s$  belongs to one of the following classes:  $(2\zeta_1, \sigma_\varrho)$ , and  $(\zeta_2, \sigma_{\varrho,2})$ .

*Proof.* We start by identifying  $\mathcal{Q}_2^s$  with  $P_e^{\sigma_e}$ .

The  $\varrho$ -semi-canonical elements of  $SU(4)$  are precisely the elements

$$2\zeta_1, 4\zeta_1, \zeta_2, 2\zeta_2, 2\zeta_1 + \zeta_2, 2\zeta_1 + 2\zeta_2, 4\zeta_1 + \zeta_2, 4\zeta_1 + 2\zeta_2.$$

By Proposition 9, all these elements correspond to the symmetric space  $\mathcal{Q}_2^s$ .

We claim that  $\exp \pi\zeta_2$  is not in the connected component

$$P_e^{\sigma_e} = \{g J_o g^{-1} J_o^{-1} | g \in SU(4)\}.$$

In fact,  $\exp(\pi\zeta_2)J_o = g J_o g^{-1} \cong g Sp(n)$  for the unitary transformation  $g$  defined by  $g(e_1) = e_4$ ,  $g(e_4) = e_1$ ,  $g(e_2) = e_3$  and  $g(e_3) = -e_2$ . Since  $\det g \neq 1$  we conclude that  $\exp \pi\zeta_2$  does not belong to  $P_e^{\sigma_e}$ . Similarly, one can check that  $\exp \pi(2\zeta_1 + \zeta_2)$  is not in  $P_e^{\sigma_e}$ .

Hence, since  $\exp \pi 2\zeta_1$  belongs to the centre of  $SU(4)$ , any harmonic map  $\varphi : S^2 \rightarrow \mathcal{Q}_2^s \cong P_e^{\sigma_e}$  belongs to one of the following classes:  $(2\zeta_1, \sigma_\varrho)$ ,  $(\zeta_2, \sigma_{\varrho,2})$ , and  $(2\zeta_1 + \zeta_2, \sigma_{\varrho,2})$ . It remains to check that, in view of Proposition 16, harmonic maps in the class  $(2\zeta_1 + \zeta_2, \sigma_{\varrho,2})$  can be normalized to harmonic maps in the class  $(\zeta_2, \sigma_{\varrho,2})$ .

It is clear that  $2\zeta_1 + \zeta_2 \preceq \zeta_2$ . On the other hand, for any positive root  $L_i - L_j \in \Delta^+$ , with  $i < j$ , we have  $(L_i - L_j)(2\zeta_1)/i \leq (L_i - L_j)(2\zeta_1 + \zeta_2)/i$ , where the equality holds in just one case:  $(L_2 - L_3)(2\zeta_1) = (L_2 - L_3)(2\zeta_1 + \zeta_2) = 2i$ . However,  $\mathfrak{g}_{L_2-L_3} \subset \mathfrak{k}_{\sigma_{\varrho,2}}$ , which means that the conditions of Proposition 16 hold for  $\zeta = 2\zeta_1 + \zeta_2$  and  $\zeta' = \zeta_2$ , and consequently harmonic maps in the class  $(2\zeta_1 + \zeta_2, \sigma_{\varrho,2})$  can be normalized to harmonic maps in the class  $(\zeta_2, \sigma_{\varrho,2})$ .  $\square$

Following the same procedure as before, one can see that any harmonic map  $\varphi \rightarrow \mathcal{Q}_2^s$  in the class  $(2\zeta_1, \sigma_\varrho)$  admits an extended solution of the form

$$\Phi = \begin{pmatrix} 1 & 0 & c_1 + a\lambda & c_2 \\ 0 & 1 & c_3 & c_1 - a\lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{2\zeta_1},$$

where  $c_1, c_2, c_3 \in \mathbb{C}$  are constants,  $a$  is a meromorphic function on  $S^2$ . The harmonic map is recovered by setting  $\varphi = \Phi_{-1} J_o$ . Reciprocally, given arbitrary complex constants  $c_1, c_2, c_3$  and a meromorphic function  $a : S^2 \rightarrow \mathbb{C}$ ,

such  $\Phi$  is an extended solution associated to some harmonic map in the class  $(2\zeta_1, \sigma_\varrho)$  (the harmonicity equations do not impose any restriction to  $a$ ).

Similarly, any harmonic map  $\varphi \rightarrow \mathcal{Q}_2^s$  in the class  $(\zeta_2, \sigma_{\varrho,2})$  admits an extended solution of the form

$$\Phi = \begin{pmatrix} 1 & b & a & c \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{\zeta_2},$$

where  $a, b$  and  $c$  are meromorphic functions satisfying  $c' = ba' - b'a$ . Since  $P_{\zeta_2}^{\sigma_{\varrho,2}} = \exp(\pi\zeta_2)P_e^{\sigma_{\varrho}}$ , the harmonic map is recovered by setting  $\varphi = \exp(\pi\zeta_2)\Phi_{-1}J_o$ .

**5.3.2. Harmonic maps into  $\mathcal{L}_{2n}^s$ .** The outer symmetric  $SU(2n)$ -space that corresponds to the involution  $\sigma_{\varrho,1}$  is the space of special Lagrangian subspaces  $\mathcal{L}_{2n}^s \cong SU(2n)/SO(2n)$ . Take as base point the Lagrangian space  $L_o = \text{Span}\{e_1, \dots, e_{2n}\}$  of  $\mathbb{R}^{4n}$  and let  $\tau_0$  be the corresponding conjugation, so that the Cartan embedding of  $\mathcal{L}_{2n}^s$  into  $SU(2n)$  is given by  $\tau = g\tau_o g^{-1} \mapsto g\tau_0 g^{-1}\tau \in P_e^{\sigma_{\varrho,1}}$ .

**Lemma 25.** For each  $\zeta \in \mathcal{I}(SU(2n)^{\sigma_{\varrho,1}})$  we have  $\exp \pi\zeta \in P_e^{\sigma_{\varrho,1}}$ .

*Proof.* Each  $\zeta \in \mathcal{I}(SU(2n)^{\sigma_{\varrho,1}})$  can be written as  $\zeta = \sum_{i=1}^n n_i(E_i - E_{2n+1-i})$ . Hence,  $\exp \pi\zeta = \pi_V - \pi_V^\perp$ , where  $V = \bigoplus_{n_i \text{ even}} \text{Span}\{e_i, e_{2n+1-i}\}$ . Define  $g \in SU(2n)$  as follows: if  $n_i$  is even, then  $g(e_i) = e_i$  and  $g(e_{2n+1-i}) = e_{2n+1-i}$ ; if  $n_i$  is odd, then  $g(e_i) = ie_i$  and  $g(e_{2n+1-i}) = -ie_{2n+1-i}$ . We have  $\exp \pi\zeta = g\tau_0 g^{-1}\tau_0$ , that is  $\exp \pi\zeta \in P_e^{\sigma_{\varrho,1}}$ .  $\square$

Now, identify  $\mathcal{L}_{2n}^s$  with  $P_e^{\sigma_{\varrho,1}}$  via its Cartan embedding. By Theorem 12, any harmonic map  $\varphi : S^2 \rightarrow P_e^{\sigma_{\varrho,1}}$  admits an extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta'}^{\sigma_{\varrho,1}}(SU(2n))$ , for some  $\zeta' \in \mathcal{I}'(SU(2n)) \cap \mathfrak{k}_{\sigma_{\varrho,1}}$  and some discrete subset  $D$ . We can assume that  $\zeta'$  is a  $\varrho$ -semi-canonical element. The corresponding  $S^1$ -invariant solution  $u_\zeta \circ \Phi$  takes values in  $\Omega_\xi(SU(2n)^{\sigma_{\varrho,1}})$ , with  $\xi \in \mathcal{I}'(SU(2n)^{\sigma_{\varrho,1}})$ ; and both  $\Phi_{-1}$  and  $(u_\zeta \circ \Phi)_{-1}$  take values in  $P_\xi^{\sigma_{\varrho,1}}$ . A priori,  $\xi$  can be different from  $\zeta$  since  $\sigma_{\varrho,1}$  is not a fundamental outer involution. However, by Lemma 25 we have  $P_\xi^{\sigma_{\varrho,1}} = P_e^{\sigma_{\varrho,1}} = P_{\zeta'}^{\sigma_{\varrho,1}}$ .

If  $\zeta$  is a  $\varrho$ -canonical element such that  $\zeta' \preceq \zeta$  and  $\mathcal{U}_{\zeta', \zeta' - \zeta}(\Phi)$  is constant, then, taking into account Proposition 15, there exists a  $T_\tau$ -invariant extended solution  $\tilde{\Phi} : S^2 \setminus D \rightarrow U_\zeta^\tau(SU(2n))$ , where

$$\tau = \text{Ad}(\exp \pi(\zeta' - \zeta)) \circ \sigma_{\varrho,1}. \quad (42)$$

such that  $\tilde{\Phi}_{-1}$  take values in  $P_\zeta^\tau$  and  $\varphi$  is given up to isometry by

$$\varphi = \exp(\zeta' - \zeta)\tilde{\Phi}_{-1}\tau_0. \quad (43)$$

We conclude that, given a pair  $(\zeta, \tau)$ , where  $\zeta \in \mathcal{I}(SU(2n)^{\sigma_{\varrho}})$  is a  $\varrho$ -canonical element and  $\tau$  is an outer involution of the form (42), any extended solution  $\tilde{\Phi} : S^2 \setminus D \rightarrow U_\zeta^\tau(SU(2n))$  gives rise via (43) to an harmonic map  $\varphi$  from the two-sphere into  $\mathcal{L}_{2n}^s$  and, conversely, all harmonic two-spheres into  $\mathcal{L}_{2n}^s$  arise in this way.

For  $\mathcal{L}_4^s$ , since  $\exp \pi 2\zeta_1$  belongs to the centre of  $SU(4)$ , we have five classes of harmonic maps into  $\mathcal{L}_4^s$ :

$$(2\zeta_1, \sigma_{\varrho,1}), (\zeta_2, \sigma_{\varrho,1}), (2\zeta_1 + \zeta_2, \sigma_{\varrho,1}), (\zeta_2, \text{Ad} \exp \pi\zeta_2 \circ \sigma_{\varrho,1}), (2\zeta_1 + \zeta_2, \text{Ad} \exp \pi\zeta_2 \circ \sigma_{\varrho,1}).$$

Let us consider in detail the class  $(\zeta_2, \sigma_{\varrho,1})$ . Clearly  $r(\zeta_2) = 2$ . Let  $W_1, W_2, W_3$  and  $W_4$  be the complex one-dimensional images of  $E_1, E_2, E_3$  and  $E_4$ , respectively. That is,  $W_i = \text{Span}\{v_i\}$ , where  $v_i$  are defined by (41). Any extended solution  $\Phi : S^2 \setminus D \rightarrow U_{\zeta_2}^{\sigma_{\varrho,1}}$  is given by  $\Phi = \exp C \cdot \gamma_{\zeta_2}$ , with  $\gamma_{\zeta_2}(\lambda) = \lambda^{-1}\pi_{W_4} + \pi_{W_3 \oplus W_2} + \lambda\pi_{W_1}$ , for some holomorphic vector-valued function  $C : S^2 \setminus D \rightarrow (\mathfrak{u}_{\zeta_2}^0)_{\sigma_{\varrho,1}}$ , where

$$(\mathfrak{u}_{\zeta_2}^0)_{\sigma_{\varrho,1}} = (\mathfrak{p}_0^{\zeta_2})^\perp \cap \mathfrak{k}_{\sigma_{\varrho,1}}^\mathbb{C} + \lambda(\mathfrak{p}_1^{\zeta_2})^\perp \cap \mathfrak{m}_{\sigma_{\varrho,1}}^\mathbb{C}$$

and

$$\begin{aligned} (\mathfrak{p}_0^{\zeta_2})^\perp \cap \mathfrak{k}_{\sigma_{\varrho,1}}^{\mathbb{C}} &= (\mathfrak{g}_{L_1-L_2} \oplus \mathfrak{g}_{L_3-L_4} \oplus \mathfrak{g}_{L_1-L_3} \oplus \mathfrak{g}_{L_2-L_4}) \cap \mathfrak{k}_{\sigma_{\varrho,1}}^{\mathbb{C}}, \\ (\mathfrak{p}_1^{\zeta_2})^\perp \cap \mathfrak{m}_{\sigma_{\varrho,1}}^{\mathbb{C}} &= \mathfrak{g}_{L_1-L_4} \cap \mathfrak{m}_{\sigma_{\varrho,1}}^{\mathbb{C}} = \mathfrak{g}_{L_1-L_4}. \end{aligned}$$

We have  $\sigma_{\varrho,1}(X_{1,2}) = -X_{3,4}$  and  $\sigma_{\varrho,1}(X_{1,3}) = X_{2,4}$ . Hence we can write  $C = C_0 + C_1\lambda$ , with

$$C_0 = a(X_{1,2} - X_{3,4}) + b(X_{1,3} + X_{2,4}), \quad C_1 = cX_{1,4}$$

for some meromorphic functions  $a, b, c$  on  $S^2$ . The harmonicity equations impose that  $ab' - ba' = 0$ , which means that  $b = \alpha a$  for some constant  $\alpha \in \mathbb{C}$ . Hence given arbitrary meromorphic functions  $a, c$  on  $S^2$  and a complex constant  $\alpha$ ,

$$\Phi = \begin{pmatrix} 1 & a & \alpha a & c\lambda \\ 0 & 1 & 0 & -\alpha a \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \gamma_{\zeta_1},$$

is an extended solution associated to some harmonic map in the class  $(\zeta_2, \sigma_{\varrho,1})$ . Reciprocally, any harmonic map in such class arises in this way.

#### REFERENCES

- [1] F. E. Burstall and M. A. Guest, *Harmonic two-spheres in compact symmetric spaces, revisited*, Math. Ann. **309** (1997), no. 4, 541–572.
- [2] F. E. Burstall, J. H. Rawnsley, *Twistor Theory for Riemannian Symmetric Spaces*, Lectures Notes in Math. 1424 Berlin, Heidelberg: 1990.
- [3] E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Diff. Geom., **1** (1967), 111–125.
- [4] N. Correia and R. Pacheco, *Harmonic maps of finite uniton number into  $G_2$* , Math. Z. **271** (2012), no. 1-2, 13–32.
- [5] N. Correia and R. Pacheco, *Extended Solutions of the Harmonic Map Equation in the Special Unitary Group*, Q. J. Math. **65** (2014), no. 2, 637–654.
- [6] N. Correia and R. Pacheco, *Harmonic maps of finite uniton number and their canonical elements*, Ann. Global Anal. Geom. **47** (2015), no. 4, 335–358.
- [7] J. Dorfmeister, F. Pedit and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), 633–668.
- [8] J.-H. Eschenburg, A.-L. Mare, and P. Quast, *Pluriharmonic maps into outer symmetric spaces and a subdivision of Weyl chambers*, Bull. London Math. Soc. **42** (2010), no. 6, 1121–1133.
- [9] J. Eells and J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Adv. in Math. **49** (1983), no. 3, 217–263.
- [10] W. Fulton and J. Harris, *Representation theory (a first course)*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
- [11] M. A. Guest and Y. Ohnita, *Loop group actions on harmonic maps and their applications*, Harmonic maps and integrable systems, 273–292, Aspects Math., E23, Vieweg, Braunschweig, 1994.
- [12] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York 1978.
- [13] H. Ma, *Explicit construction of harmonic two-spheres in  $SU(3)/SO(3)$* , Kyushu J. Math. **55** (2001), 237–247.
- [14] S. Murakami, *Sur la classification des algèbres de Lie réelles et simples*, Osaka J. Math. **2** (1965), 291–307.
- [15] A.N. Pressley and G.B. Segal, *Loop Groups*, Oxford University Press, 1986.
- [16] A. Bahy-El-Dien and J.C. Wood, *The explicit construction of all harmonic two-spheres in  $G_2(\mathbb{R}^n)$* , J. Reine Angew. Math. **398** (1989), 36–66.
- [17] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.
- [18] W. Ziller, *Lie groups, representation theory and symmetric spaces*. Notes for a course given in the fall of 2010 at the University of Pennsylvania and 2012 at IMPA.

UNIVERSIDADE DA BEIRA INTERIOR, RUA MARQUÊS D'ÁVILA E BOLAMA, 6200-001 COVILHÃ, PORTUGAL  
E-mail address: ncorreia@ubi.pt, rpacheco@ubi.pt